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EXTREME POINTS OF A CLASS OF SUBORDINATE FUNCTIONS

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ABSTRACT. It is shown that, if $F(z)$ is subordinate to $H((1+z)/(1-z))$ in the unit disc, where $H(w)$ is a quadratic polynomial for which $H'(w) \neq 0$ in the right half-plane, then

$$F(z) = \int_T H\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right) d\mu$$

for a suitable probability measure μ on the unit circle T.

In [**1**, Problem 5.59], the problem was raised of describing functions $G(z)$ analytic in $U = \{|z| < 1\}$ with the property that the extreme points of the closed convex hull of the class

$$S_G = \{F: F \prec G\} \tag{1}$$

had the form $G(ze^{it})$ $(0 \leqslant t < 2\pi)$. Here $F \prec G$ means F subordinate to G. It is easily seen that the functions $G(ze^{it})$ are extreme points. The general problem of describing support points and extreme points of subordination families has recently been taken up by Hallenbeck and MacGregor [**2**]. In this note we will establish a result, which we claimed without proof in [**1**]. The method of proof is one which may generalise to other classes.

THEOREM. *Let $G(z)$ be a function of the form $H((1+z)/(1-z))$, where $H(w)$ is a univalent, quadratic polynomial in $\{\operatorname{Re} w > 0\}$. Then the extreme points of the closed convex hull of S_G have the form $G(ze^{it})$ $(0 \leqslant t < 2\pi)$.*

PROOF. Let $H(w) = A_0 + A_1 w + A_2 w^2$. The condition for univalence in $\{\operatorname{Re} w > 0\}$ is

$$\operatorname{Re}(A_1/A_2) \geqslant 0. \tag{2}$$

Thus

$$\begin{aligned} G(z) &= A_0 - A_1 + A_2 + 2A_1\frac{1}{1-z} + 4A_2\frac{z}{(1-z)^2} \\ &= A_0 + A_1 + A_2 + \sum_{n=1}^{\infty} 2(A_1 + 2nA_2)z^n. \end{aligned} \tag{3}$$

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Without loss of generality we may assume $A_0 + A_1 + A_2 \neq 0$. Then

$$|A_1 + 2nA_2| \geqslant 2|A_2| \tag{4}$$

and so the convolution inverse

$$G_i(z) = \frac{1}{A_0 + A_1 + A_2} + \sum_{n=1}^{\infty} \frac{1}{2(A_1 + 2nA_2)} z^n \tag{5}$$

is analytic in U. For $F \in S_G$ we show that

$$\operatorname{Re}(G_i * F)(z) > 1/2 \qquad (z \in U). \tag{6}$$

Writing $P = G_i * F$ we have $P(0) = 1$, so (6) implies

$$F(z) = (G*P)(z) = \frac{1}{2\pi}\int_0^{2\pi} G(ze^{it})\, d\mu(t), \tag{7}$$

where $\mu(t)$ is a nondecreasing function on $[0, 2\pi]$ with $\mu(2\pi) = \mu(0) + 2\pi$. Here we have applied the Herglotz representation for P. The theorem clearly follows from (7). Now if (6) does not hold, we deduce from the Clunie-Jack lemma [**3**] that $\exists \xi \in U$, c real with

$$P(\xi) = \tfrac{1}{2} + ic, \quad \xi P'(\xi) = -d \leqslant -(\tfrac{1}{4} + c^2). \tag{8}$$

Then

$$\begin{aligned} F(\xi) &= (G*P)(\xi) = A_0 - A_1 + A_2 + 2A_1P(\xi) + 4A_2\xi P'(\xi) \\ &= A_0 + A_1(2ic) + A_2(-4c^2) - 4\alpha A_2 = H(2ic) - 4\alpha A_2, \end{aligned} \tag{9}$$

where $\alpha \geqslant 0$. But

$$F(\xi) = H(w), \tag{10}$$

where $\operatorname{Re} w > 0$. Hence the nonpositive real number

$$\begin{aligned} -4\alpha &= \frac{F(\xi) - H(2ic)}{A_2} = \frac{A_1}{A_2}(w - 2ic) + w^2 - (2ic)^2 \\ &= (w - 2ic)\left(\frac{A_1}{A_2} + w + 2ic\right) \end{aligned} \tag{11}$$

is the product of two numbers of positive real part, which is impossible. Therefore (6) holds and the proof is complete.

References

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