PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 1, May 1984

APPROXIMATE UNITARY EQUIVALENCE OF POWER PARTIAL ISOMETRIES

KENNETH R. DAVIDSON¹

ABSTRACT. Every power partial isometry (p.p.i.) in the Calkin algebra lifts to a p.p.i. in $\mathcal{B}(\mathcal{H})$.

An element $u$ in a $\mathcal{C}^*$ algebra is a power partial isometry (p.p.i.) if $u^n$ is a partial isometry for every integer $n \geq 1$. This notion was introduced by Halmos and Wallen [5], and they characterized all p.p.i.'s in $\mathcal{B}(\mathcal{H})$. In a penetrating study [6], Herrero classifies p.p.i.'s up to their unitary orbits and up to unitary equivalence modulo the compacts. In this note, we show that every p.p.i. in the Calkin algebra lifts to a p.p.i. in $\mathcal{B}(\mathcal{H})$. This is done by using a few straightforward computations involving the theory of $\mathcal{C}^*$ extensions. Some of the results of [6] follow from this method as well.

In this paper, Hilbert spaces are always separable. The ideal of compact operators will be denoted by $\mathcal{K}$. The canonical quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ is denoted by $\pi$. Two elements $s$ and $t$ of the Calkin algebra are (strongly) compalent ($s \sim t$) if there is a unitary $U$ in $\mathcal{B}(\mathcal{H})$ so that $t = \pi(U)s\pi(U^*)$. They are weakly compalent ($s =_{\omega} t$) if there is a unitary $u$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}$ so that $t = usu^*$. For p.p.i.'s, these two notions may differ. Given a nuclear $\mathcal{C}^*$ algebra $\mathcal{A}$, Ext $\mathcal{A}$ denotes the group of extensions of $\mathcal{K}$ by $\mathcal{A}$ modulo compalence [1, 3, 8] and Ext$_{\omega} \mathcal{A}$ denotes the quotient of Ext $\mathcal{A}$ modulo weak compalence. We use little more than the fact that when

$$0 \rightarrow J \rightarrow \mathcal{A} \rightarrow \mathcal{A}/J \rightarrow 0$$

is an exact sequence, and Ext $J = 0 =$ Ext $\mathcal{A}/J$, then Ext $\mathcal{A} = 0$.

The first step is to classify p.p.i.'s up to algebraic equivalence by computing $\mathcal{C}^*(T)$ for all p.p.i.'s. That is, given two p.p.i.'s $S$ and $T$, when is there a $\mathcal{C}^*$ isomorphism $\phi$ of $\mathcal{C}^*(S)$ onto $\mathcal{C}^*(T)$ such that $\phi(S) = T$? For convenience, this will be written $S \sim T$. Let $N_k$ be the nilpotent Jordan cell of order $k$ acting on a $k$-dimensional Hilbert space, and let $S$ be the unilateral shift. If $\alpha$ belongs to $N_0 \cup \{\omega\}$, let $A^{(\alpha)}$ be the direct sum of $\alpha$ copies of $\mathcal{A}$ acting on $\mathcal{H}^{(\alpha)}$, the direct sum of $\alpha$ copies of $\mathcal{H}$. By [5], every p.p.i. $T$ represented on a Hilbert space can be decomposed as

$$T \cong \bigoplus_{k=1}^{\infty} N_k^{(\alpha_k)} \oplus S^{(\alpha)} \oplus S^{(\beta)} \oplus V$$

where $V$ is a unitary. (If $\alpha_k = 0$, the summand is deemed to be absent.) The existence of the finite nilpotent blocks is easy to determine because there is a

¹Research partially supported by NSERC grant #A3488. I would also like to thank the people at Indiana University for their hospitality during the period when this work was completed.

© 1984 American Mathematical Society
0002-9939/84 $1.00 + .25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
polynomial \( p_k(x,y) \) in two noncommuting variables such that \( P_k = p_k(T,T^\ast) \) is a (selfadjoint) projection onto the summand \( 2^{(\alpha_k)} \) on which \( N_k^{(\alpha_k)} \) acts. The classification in [5] proceeds by pulling off all the nilpotent blocks, then one removes the isometric and coisometric parts, and what remains is the unitary.

This procedure is not possible in an arbitrary C* algebra because one can pull off only finitely many pieces. Also, if \( T \) has no nilpotent blocks and is not unitary, then \( C^*(T) \) has only a few possibilities up to algebraic equivalence. If \( \Sigma \) is a nonempty compact subset of the unit circle \( S^1 \), let \( V_{\Sigma} \) be a fixed unitary with spectrum \( \Sigma \).

**Proposition 1.** (a) If \( T \) is a p.p.i. with no nilpotent blocks, then there are only the following possibilities:

1. \( T \) is unitary (\( TT^\ast = I = T^\ast T \)), in which case the spectrum \( \Sigma = \sigma(T) \) determines \( C^*(T) \) up to algebraic equivalence: \( T \sim V_{\Sigma} \).
2. \( T \) is a nonunitary isometry (\( T^\ast T = I \neq TT^\ast \)), in which case \( T \sim S \).
3. \( T \) is a nonunitary coisometry (\( T^\ast T \neq I = TT^\ast \)), in which case \( T \sim S^\ast \).
4. \( T \) is neither isometric nor coisometric, in which case \( T \sim S \oplus S^\ast \).

(b) If \( T \) is a p.p.i. with finitely many nilpotent blocks of orders \( k_1, \ldots, k_n \), then \( P_0 = I - \sum_{i=1}^n P_{k_i} \) is a projection in \( C^*(T) \) and \( T_1 = P_0 T \) is determined up to algebraic equivalence by (a). So \( T \sim \bigoplus_{i=1}^n N_{k_i} \oplus T_1 \).

(c) If \( T \) is a p.p.i. with nilpotent blocks of orders \( k_1 < k_2 < k_3 < \cdots \), then \( T \sim \bigoplus_{i=1}^{\infty} N_{k_i} \).

**Proof.** If \( T \) is a p.p.i. in a C* algebra \( \mathcal{A} \), represent \( \mathcal{A} \) on a Hilbert space and apply the theorem of [5]. Most of this proposition is straightforward, based on the simple fact that for any representation \( \pi \) of \( C^*(T) \), one has \( T \sim T \oplus \pi(T) \).

Case (a1) follows from the spectral theorem. Since there is a *homomorphism of \( C^*(S) \) onto \( C(S^1) \) taking \( S \) to the identity function \( z \), one has \( S \sim S^\alpha \oplus V \) for any \( \alpha \geq 1 \) and any unitary \( V \). Case (a3) is similar. Finally, the same remarks show that \( S \oplus S^* \sim S^\alpha \oplus S^\beta \oplus V \) for any \( \alpha \geq 1, \beta \geq 1 \) and any unitary \( V \). The projections \( P_k \) onto the nilpotent part of order \( k \) belong to \( C^*(T) \). Thus if there are only finitely many \( P_{k_i} \)'s which are not zero, one can define \( P_0 = I - \sum_{i=1}^n P_{k_i} \). This pulls out a direct summand \( T_1 \) which is analyzed by part (a).

Given an infinite sequence \( k_1 < k_2 < k_3 < \cdots \), consider the p.p.i. \( W = \bigoplus_{i=1}^{\infty} N_{k_i} \), acting on \( \mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i \). Let \( J \) be the ideal \( C^*(W) \cap K \), where \( K \) is the compact operators. Clearly \( J \) contains the projection \( P_{k_i} \) onto \( \mathcal{H}_i \), and thus contains all of \( \mathcal{B}(\mathcal{H}_i) \cong M_{k_i}(\mathbb{C}) \) for each \( i \geq 1 \). It follows that \( J \) is the c_0 direct sum \( \bigoplus_{i=1}^{\infty} \mathcal{B}(\mathcal{H}_i) \). The quotient \( C^*(W)/J \) is isomorphic to \( C^*(\mathfrak{w}) \) where \( \mathfrak{w} \) is the image of \( W \) in the Calkin algebra. Because \( P_k \) is compact, \( p_k(\mathfrak{w}) = \pi P_k = 0 \) so \( \mathfrak{w} \) is a p.p.i. with no nilpotent summands. Also,

\[
\mathfrak{w}^* \mathfrak{w}^* = \pi W W^* \neq \mathfrak{w}^* \mathfrak{w} \mathfrak{w}^* \mathfrak{w} \neq 1,
\]

so \( \mathfrak{w} \) is neither isometric nor coisometric. By (a), \( \mathfrak{w} \sim S \oplus S^* \). Thus by (a) and (b),

\[
W = \bigoplus_{i=1}^{\infty} N_{k_i}^{(\alpha_i)} \oplus S^{(\alpha)} \oplus S^{(\beta)} \oplus V
\]

for any \( \alpha_i \geq 1 \), any \( \alpha, \beta \) and any unitary \( V \). \( \square \)

**Theorem 2.** Every p.p.i. \( t \) in the Calkin algebra lifts to a p.p.i. in \( \mathcal{B}(\mathcal{H}) \). These p.p.i. 's are classified up to unitary equivalence in the Calkin algebra as follows.

(a1) If \( t \) is unitary and \( \sigma(t) = \Sigma \) is a proper subset of \( S^1 \), then \( t \sim \pi(V_Q^{(\infty)}) \).
(a2) If \( t \) is unitary and \( \sigma(t) = S^1 \), and \( \text{ind} \ t = n \), then \( t \cong \pi(S^{(-n)}) \) if \( n < 0 \), \( t \cong \pi(V_{S^1}) \) if \( n = 0 \), and \( t \cong \pi(S^{(+n)}) \) if \( n > 0 \).

(b) If \( t \) is isometric but not unitary, \( t \cong \pi(S^{(\infty)}) \).

(c) If \( t \) is coisometric but not unitary, \( t \cong \pi(S^{*(-\infty)}) \).

(d) If \( t \) has no nilpotent part and is neither isometric nor coisometric, then \( t \cong \pi(S^{(\infty)} \oplus S^{*(-\infty)}) \).

(e) If \( t \) has finitely many nilpotent parts \( k_1, \ldots, k_n \) with corresponding projections \( p_1, \ldots, p_n \), then

\[
\begin{align*}
(e1-4) & \text{ if } p_0 = 1 - \sum_{i=1}^n p_i \text{ is not zero, then } t \cong \pi(\bigoplus_{i=1}^n N_{k_i}^{(\infty)} \oplus T_1) \text{ where } T_1 \text{ is determined by (a)-(d),} \\
(e5) & \text{ if } p = 0, \text{ then there is an integer } d \text{ with } 0 \leq d < \gcd\{k_1, \ldots, k_n\} \text{ such that } t \cong \pi(\bigoplus_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d) \text{ where } \mathcal{O}_d = N_1^{(d)} \text{ is the zero operator on a } d\text{-dimensional space.}
\end{align*}
\]

(f) If \( t \) has nilpotent parts for the infinite sequence \( k_1 < k_2 < k_3 < \cdots \), then \( t \cong \pi(\bigoplus_{i=1}^n N_{k_i}^{(\infty)}) \).

Proof. Cases (a1) and (a2) follow from early results of [2]. Cases (b) and (c) are well known [4]. This is also immediate from the fact that \( \text{Ext}(C^*(S)) = 0 \). For case (d), we have \( t \sim S \oplus S^* \sim \pi(S^{(\infty)} \oplus S^{*(-\infty)}) \). It suffices to show that \( \text{Ext}(C^*(S \oplus S^*)) = 0 \). Let \( T = S \oplus S^* \), and let \( J \) be the ideal in \( C^*(T) \) generated by \( I - T^*T \). It is easy to see that \( J \) is isomorphic to the compact operators \( K \), and that \( C^*(T)/J \) is isomorphic to \( C^*(S) \). Thus

\[
0 \rightarrow K \rightarrow C^*(T) \rightarrow C^*(S) \rightarrow 0
\]

is exact. But \( \text{Ext} K = 0 = \text{Ext} C^*(S) \), and hence \( \text{Ext} C^*(T) = 0 \). So \( t \) is unitarily equivalent to \( \pi(S^{(\infty)} \oplus S^{*(-\infty)}) \).

In case (e5), \( C^*(t) \) is isomorphic to \( \bigoplus_{i=1}^n M_{k_i}(C) \), and thus \( \text{Ext} C^*(t) \) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) where \( p = \gcd\{k_1, \ldots, k_n\} \) and \( \text{Ext}_w C^*(t) = 0 \). It follows readily that if \( d \) is the integer with \( 0 \leq d < p \) such that \( d + p\mathbb{Z} \) is the element of \( \text{Ext} C^*(t) \) determined by \( t \), then \( t \cong \pi(\bigoplus_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d) \), and \( t \cong \pi(\bigoplus_{i=1}^n N_{k_i}^{(\infty)}) \).

When \( p_0 \neq 0 \), let \( t = t_0 \oplus t_1 \) where \( p_0 = 0 \oplus 1 \). The above analysis shows that

\[
t \cong \pi\left(\bigoplus_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d \oplus T_1\right) \cong \pi\left(\bigoplus_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d \oplus T_1\right).
\]

The latter term has no nilpotent part, so it is analyzed by parts (a) and (b).

Finally, if \( t \) has nilpotent parts for the infinite sequence \( k_1 < k_2 < k_3 < \cdots \), we refer back to the analysis of Proposition 1 to obtain an ideal \( J \) isomorphic to \( \bigoplus_{i=1}^\infty M_{k_i}(C) \) in \( C^*(t) \) so that

\[
0 \rightarrow J \rightarrow C^*(t) \rightarrow C^*(S \oplus S^*) \rightarrow 0.
\]

By [7], \( \text{Ext} J = 0 \). Since \( \text{Ext} C^*(S \oplus S^*) = 0 \), it follows that \( \text{Ext} C^*(t) = 0 \). Thus \( t \) is unitarily equivalent to \( \pi(\bigoplus_{i=1}^\infty N_{k_i}^{(\infty)}) \).

Remarks. (1) One of the most surprising facts, noted first in [5], is that \( \bigoplus_{k=1}^\infty N_k \) is approximately unitarily equivalent to \( S^{(\infty)} \oplus S^{*(-\infty)} \). This is now "explained" by noting that, in the Calkin algebra, both have nonisometric, non-coisometric images without nilpotent parts, and hence the images are algebraically equivalent to \( S \oplus S^* \). This can be represented in the Calkin algebra in only one way.
(2) Except in case (e5) in which $C^*(t)$ is finite dimensional, weak and strong equivalence coincide for p.p.i.'s.

(3) The classification of p.p.i.'s in $\mathcal{B}(\mathcal{H})$ up to approximate unitary equivalence is carried out in [6]. These results are also immediate from Theorem 2. In fact, it would be sufficient to show every p.p.i. in the Calkin algebra lifts to a p.p.i. without classifying them as one goes along. But the nature of our method forces us to do it all.

REFERENCES


DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO N2L 3G1, CANADA