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APPROXIMATE UNITARY EQUIVALENCE OF POWER PARTIAL ISOMETRIES

KENNETH R. DAVIDSON[1]

ABSTRACT. Every power partial isometry (p.p.i.) in the Calkin algebra lifts to a p.p.i. in $\mathcal{B}(\mathcal{H})$.

An element $\mathfrak{u}$ in a C^* algebra is a power partial isometry (p.p.i.) if $\mathfrak{u}^n$ is a partial isometry for every integer $n \geq 1$. This notion was introduced by Halmos and Wallen [**5**], and they characterized all p.p.i.'s in $\mathcal{B}(\mathcal{H})$. In a penetrating study [**6**], Herrero classifies p.p.i.'s up to their unitary orbits and up to unitary equivalence modulo the compacts. In this note, we show that every p.p.i. in the Calkin algebra lifts to a p.p.i. in $\mathcal{B}(\mathcal{H})$. This is done by using a few straightforward computations involving the theory of C^* extensions. Some of the results of [**6**] follow from this method as well.

In this paper, Hilbert spaces are always separable. The ideal of compact operators will be denoted by $\mathcal{K}$. The canonical quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ is denoted by π. Two elements $\mathfrak{s}$ and $\mathfrak{t}$ of the Calkin algebra are (strongly) compalent ($\mathfrak{s} \cong \mathfrak{t}$) if there is a unitary U in $\mathcal{B}(\mathcal{H})$ so that $\mathfrak{t} = \pi(U)\mathfrak{s}\pi(U^*)$. They are weakly compalent ($\mathfrak{s} \cong_w \mathfrak{t}$) if there is a unitary $\mathfrak{u}$ in $\mathcal{B}(\mathcal{H})/\mathcal{K}$ so that $\mathfrak{t} = \mathfrak{u}\mathfrak{s}\mathfrak{u}^*$. For p.p.i.'s, these two notions may differ. Given a nuclear C^* algebra $\mathcal{A}$, $\operatorname{Ext}\mathcal{A}$ denotes the group of extensions of $\mathcal{K}$ by $\mathcal{A}$ modulo compalence [**1, 3, 8**] and $\operatorname{Ext}_w \mathcal{A}$ denotes the quotient of $\operatorname{Ext}\mathcal{A}$ modulo weak compalence. We use little more than the fact that when

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{A}/\mathcal{J} \to 0$$

is an exact sequence, and $\operatorname{Ext}\mathcal{J} = 0 = \operatorname{Ext}\mathcal{A}/\mathcal{J}$, then $\operatorname{Ext}\mathcal{A} = 0$.

The first step is to classify p.p.i.'s up to algebraic equivalence by computing $C^*(T)$ for all p.p.i.'s. That is, given two p.p.i.'s S and T, when is there a C^* isomorphism ϕ of $C^*(S)$ onto $C^*(T)$ such that $\phi(S) = T$? For convenience, this will be written $S \sim T$. Let N_k be the nilpotent Jordan cell of order k acting on a k-dimensional Hilbert space, and let S be the unilateral shift. If α belongs to $\mathbf{N}_0 \cup \{\omega\}$, let $A^{(\alpha)}$ be the direct sum of α copies of A acting on $\mathcal{H}^{(\alpha)}$, the direct sum of α copies of $\mathcal{H}$. By [**5**], every p.p.i. T represented on a Hilbert space can be decomposed as

$$T \cong \bigoplus \sum_{k=1}^{\infty} N_k^{(\alpha_k)} \oplus S^{(\alpha)} \oplus S^{*(\beta)} \oplus V$$

where V is a unitary. (If $\alpha_k = 0$, the summand is deemed to be absent.) The existence of the finite nilpotent blocks is easy to determine because there is a

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polynomial $p_k(x,y)$ in two noncommuting variables such that $P_k = p_k(T,T^*)$ is a (selfadjoint) projection onto the summand $\mathcal{H}_k^{(\alpha_k)}$ on which $N_k^{(\alpha_k)}$ acts. The classification in [5] proceeds by pulling off all the nilpotent blocks, then one removes the isometric and coisometric parts, and what remains is the unitary.

This procedure is not possible in an arbitrary C^* algebra because one can pull off only finitely many pieces. Also, if T has no nilpotent blocks and is not unitary, then $C^*(T)$ has only a few possibilities up to algebraic equivalence. If Σ is a nonempty compact subset of the unit circle S^1, let V_Σ be a fixed unitary with spectrum Σ.

PROPOSITION 1. (a) *If T is a p.p.i. with no nilpotent blocks, then there are only the following possibilities*:

(a1) *T is unitary ($TT^* = I = T^*T$), in which case the spectrum $\Sigma = \sigma(T)$ determines $C^*(T)$ up to algebraic equivalence: $T \sim V_\Sigma$.*

(a2) *T is a nonunitary isometry ($T^*T = I \neq TT^*$), in which case $T \sim S$.*

(a3) *T is a nonunitary coisometry ($T^*T \neq I = TT^*$), in which case $T \sim S^*$.*

(a4) *T is neither isometric nor coisometric, in which case $T \sim S \oplus S^*$.*

(b) *If T is a p.p.i. with finitely many nilpotent blocks of orders $k_1, \dots, k_n$, then $P_0 = I - \sum_{i=1}^n P_{k_i}$ is a projection in $C^*(T)$ and $T_1 = P_0 T \mid P_0$ is determined up to algebraic equivalence by* (a). *So $T \sim \bigoplus \sum_{i=1}^n N_{k_i} \oplus T_1$.*

(c) *If T is a p.p.i. with nilpotent blocks of orders $k_1 < k_2 < k_3 < \cdots$, then $T \sim \bigoplus \sum_{i=1}^\infty N_{k_i}$.*

PROOF. If T is a p.p.i. in a C^* algebra $\mathcal{A}$, represent $\mathcal{A}$ on a Hilbert space and apply the theorem of [5]. Most of this proposition is straightforward, based on the simple fact that for any representation π of $C^*(T)$, one has $T \sim T \oplus \pi(T)$.

Case (a1) follows from the spectral theorem. Since there is a $*$homomorphism of $C^*(S)$ onto $C(S^1)$ taking S to the identity function z, one has $S \sim S^{(\alpha)} \oplus V$ for any $\alpha \geq 1$ and any unitary V. Case (a3) is similar. Finally, the same remarks show that $S \oplus S^* \sim S^{(\alpha)} \oplus S^{*(\beta)} \oplus V$ for any $\alpha \geq 1$, $\beta \geq 1$ and any unitary V. The projections P_k onto the nilpotent part of order k belong to $C^*(T)$. Thus if there are only finitely many P_{k_i}'s which are not zero, one can define $P_0 = I - \sum_{i=1}^n P_{k_i}$. This pulls out a direct summand T_1 which is analyzed by part (a).

Given an infinite sequence $k_1 < k_2 < k_3 < \cdots$, consider the p.p.i. $W = \bigoplus \sum_{i=1}^\infty N_{k_i}$ acting on $\mathcal{H} = \bigoplus \sum_{i=1}^\infty \mathcal{H}_i$. Let $\mathcal{J}$ be the ideal $C^*(W) \cap \mathcal{K}$, where $\mathcal{K}$ is the compact operators. Clearly $\mathcal{J}$ contains the projection P_{k_i} onto $\mathcal{H}_i$, and thus contains all of $\mathcal{B}(\mathcal{H}_i) \cong M_{k_i}(\mathbf{C})$ for each $i \geq 1$. It follows that $\mathcal{J}$ is the c_0 direct sum $\bigoplus \sum_{i=1}^\infty \mathcal{B}(\mathcal{H}_i)$. The quotient $C^*(W)/\mathcal{J}$ is isomorphic to $C^*(\mathfrak{w})$ where $\mathfrak{w}$ is the image of W in the Calkin algebra. Because P_k is compact, $p_k(\mathfrak{w}) = \pi P_k = 0$ so $\mathfrak{w}$ is a p.p.i. with no nilpotent summands. Also,

$$\mathfrak{w}\mathfrak{w}^* = \pi W W^* \neq 1 \quad \text{and} \quad \mathfrak{w}^*\mathfrak{w} = \pi W^* W \neq 1,$$

so $\mathfrak{w}$ is neither isometric nor coisometric. By (a), $\mathfrak{w} \sim S \oplus S^*$. Thus by (a) and (b),

$$W \sim \bigoplus \sum_{i=1} N_{k_i}^{(\alpha_i)} \oplus S^{(\alpha)} \oplus S^{*(\beta)} \oplus V$$

for any $\alpha_i \geq 1$, any α, β and any unitary V. $\square$

THEOREM 2. *Every p.p.i. $\mathfrak{t}$ in the Calkin algebra lifts to a p.p.i. in $\mathcal{B}(\mathcal{H})$. These p.p.i.'s are classified up to unitary equivalence in the Calkin algebra as follows.*

(a1) *If $\mathfrak{t}$ is unitary and $\sigma(t) = \Sigma$ is a proper subset of S^1, then $\mathfrak{t} \cong \pi(V_\Sigma^{(\infty)})$.*

(a2) *If* $\mathfrak{t}$ *is unitary and* $\sigma(\mathfrak{t}) = S^1$, *and* $\operatorname{ind}\mathfrak{t} = n$, *then* $\mathfrak{t} \cong \pi(S^{(-n)})$ *if* $n < 0$, $\mathfrak{t} \cong \pi(V_{S_1})$ *if* $n = 0$, *and* $\mathfrak{t} \cong \pi(S^{*(n)})$ *if* $n > 0$.

(b) *If* $\mathfrak{t}$ *is isometric but not unitary,* $\mathfrak{t} \cong \pi(S^{(\infty)})$.

(c) *If* $\mathfrak{t}$ *is coisometric but not unitary,* $\mathfrak{t} \cong \pi(S^{*(\infty)})$.

(d) *If* $\mathfrak{t}$ *has no nilpotent part and is neither isometric nor coisometric, then* $\mathfrak{t} \cong \pi(S^{(\infty)} \oplus S^{*(\infty)})$.

(e) *If* $\mathfrak{t}$ *has finitely many nilpotent parts* $k_1, \ldots, k_n$ *with corresponding projections* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, *then*

(e1–4) *if* $\mathfrak{p}_0 = 1 - \sum_{i=1}^n \mathfrak{p}_i$ *is not zero, then* $\mathfrak{t} \cong \pi(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)} \oplus T_1)$ *where* T_1 *is determined by* (a)–(d),

(e5) *if* $\mathfrak{p} = 0$, *then there is an integer* d *with* $0 \le d < \text{g.c.d.}\{k_1, \ldots, k_n\}$ *such that* $\mathfrak{t} \cong \pi(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d)$ *where* $\mathcal{O}_d = N_1^{(d)}$ *is the zero operator on a* d*-dimensional space.*

(f) *If* $\mathfrak{t}$ *has nilpotent parts for the infinite sequence* $k_1 < k_2 < k_3 < \cdots$, *then* $\mathfrak{t} \cong \pi(\bigoplus \sum_{i=1}^\infty N_{k_i}^{(\infty)})$.

PROOF. Cases (a1) and (a2) follow from early results of [**2**]. Cases (b) and (c) are well known [**4**]. This is also immediate from the fact that $\operatorname{Ext}(C^*(S)) = 0$. For case (d), we have $\mathfrak{t} \sim S \oplus S^* \sim \pi(S^{(\infty)} \oplus S^{*(\infty)})$. It suffices to show that $\operatorname{Ext}(C^*(S \oplus S^*)) = 0$. Let $T = S \oplus S^*$, and let $\mathcal{J}$ be the ideal in $C^*(T)$ generated by $I - T^*T$. It is easy to see that $\mathcal{J}$ is isomorphic to the compact operators $\mathcal{K}$, and that $C^*(T)/\mathcal{J}$ is isomorphic to $C^*(S)$. Thus

$$0 \to \mathcal{K} \to C^*(T) \to C^*(S) \to 0$$

is exact. But $\operatorname{Ext}\mathcal{K} = 0 = \operatorname{Ext} C^*(S)$, and hence $\operatorname{Ext} C^*(T) = 0$. So $\mathfrak{t}$ is unitarily equivalent to $\pi(S^{(\infty)} \oplus S^{*(\infty)})$.

In case (e5), $C^*(\mathfrak{t})$ is isomorphic to $\bigoplus \sum_{i=1}^n M_{k_i}(\mathbf{C})$, and thus $\operatorname{Ext} C^*(\mathfrak{t})$ is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ where $p = \text{g.c.d.}\{k_1, \ldots, k_n\}$ and $\operatorname{Ext}_w C^*(\mathfrak{t}) = 0$ [**7**]. It follows readily that if d is the integer with $0 \le d < p$ such that $d + p\mathbf{Z}$ is the element of $\operatorname{Ext} C^*(\mathfrak{t})$ determined by $\mathfrak{t}$, then $t \cong \pi(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d)$, and $\mathfrak{t} \cong_w \pi(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)})$.

When $\mathfrak{p}_0 \ne 0$, let $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ where $p_0 = 0 \oplus 1$. The above analysis shows that

$$\mathfrak{t} \cong \pi\left(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)} \oplus \mathcal{O}_d \oplus T_1\right) \cong \pi\left(\bigoplus \sum_{i=1}^n N_{k_i}^{(\infty)}\right) \oplus \pi(\mathcal{O}_d \oplus T_1).$$

The latter term has no nilpotent part, so it is analyzed by parts (a) and (b).

Finally, if $\mathfrak{t}$ has nilpotent parts for the infinite sequence $k_1 < k_2 < k_3 < \cdots$, we refer back to the analysis of Proposition 1 to obtain an ideal $\mathcal{J}$ isomorphic to $\bigoplus \sum_{i=1}^\infty M_{k_i}(\mathbf{C})$ in $C^*(\mathfrak{t})$ so that

$$0 \to \mathcal{J} \to C^*(\mathfrak{t}) \to C^*(S \oplus S^*) \to 0.$$

By [**7**], $\operatorname{Ext}\mathcal{J} = 0$. Since $\operatorname{Ext} C^*(S \oplus S^*) = 0$, it follows that $\operatorname{Ext} C^*(\mathfrak{t}) = 0$. Thus $\mathfrak{t}$ is unitarily equivalent to $\pi(\bigoplus \sum_{i=1}^\infty N_{k_i}^{(\infty)})$. □

REMARKS. (1) One of the most surprising facts, noted first in [**5**], is that $\bigoplus \sum_{k=1}^\infty N_k$ is approximately unitarily equivalent to $S^{(\infty)} \oplus S^{*(\infty)}$. This is now "explained" by noting that, in the Calkin algebra, both have nonisometric, noncoisometric images without nilpotent parts, and hence the images are algebraically equivalent to $S \oplus S^*$. This can be represented in the Calkin algebra in only one way.

(2) Except in case (e5) in which $C^*(\mathrm{t})$ is finite dimensional, weak and strong equivalence coincide for p.p.i.'s.

(3) The classification of p.p.i.'s in $\mathcal{B}(\mathcal{H})$ up to approximate unitary equivalence is carried out in [**6**]. These results are also immediate from Theorem 2. In fact, it would be sufficient to show every p.p.i. in the Calkin algebra lifts to a p.p.i. without classifying them as one goes along. But the nature of our method forces us to do it all.

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada