PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 1, May 1984

THE LORENTZ SPACE AS A DUAL SPACE

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ABSTRACT. If (X, S, μ) is a finite, completely nonatomic measure space and $\phi(t) = t^{1/p}$ $(p > 1)$ then the Lorentz space N_ϕ is the dual space of the closed span of simple functions in M_ϕ $(= N_\phi^*)$.

In this note we characterize the closed span of simple functions in M_ϕ and show that the Lorentz space N_ϕ is its dual. Though this result was inspired by the characterization of H_1 as the dual of VMO, on account of the total lack of analyticity in the Lorentz space, the two methods have little in common.

Notation. Suppose (X, S, μ) is a completely nonatomic finite measure space, $\phi(t) = t^{1/p}$ $(p > 1)$ and $N_\phi = \{f,\ f \text{ measurable}, \int_0^\infty \phi(\mu(F_y))\,dy < \infty\}$, where $F_y = \{x,\ |f(x)| > y\}$ for $y > 0$. It is known that the dual space of N_ϕ is

$$M_\phi = \left\{ f,\ f \text{ measurable},\ \sup_{0<\mu(E)} \frac{1}{\phi(\mu(E))} \int_E |f(t)|\,d\mu < \infty \right\}.$$

For more details, see [**2**].

Define

$$M_\phi^0 = \left\{ f \in M_\phi,\ \lim_{\mu(E)\to 0} \frac{1}{\phi(\mu(E))} \int_E |f(t)|\,d\mu = 0 \right\}.$$

LEMMA 1. $M_\phi^0 =$ *closed span of simple functions in* M_ϕ.

PROOF. If $0 < \mu(E)$ and $f = \psi_E$ then

$$\sup_F \frac{1}{\mu(F)^{1/p}} \int_P |f|\,d\mu = \sup_F \frac{\mu(F \cap E)}{\mu(F)^{1/p}} \le \sup_F \mu(F \cap E)^{1/q} \le \mu(E)^{1/q},$$

where $1/p + 1/q = 1$. Taking $F = E$ we see that $\|f\|_{M_\phi} = \mu(E)^{1/q}$. Also

$$\frac{1}{\mu(F)^{1/p}} \int_F |f|\,d\mu = \frac{\mu(F \cap E)}{\mu(F)^{1/p}} \le \mu(F)^{1/q}.$$

Hence

$$\lim_{\mu(F)\to 0} \frac{1}{\phi(\mu(F))} \int_F |f|\,d\mu = 0.$$

It is easy to see that M_ϕ^0 is a closed linear subspace of M_ϕ and hence contains simple functions.

Conversely suppose that $f \in M_\phi^0$. In particular $f \in M_\phi$ and hence $f \in L_1$ [**2**, Theorems 5.6, 5.7]. Let $\epsilon > 0$. It follows by a standard approximation argument and the fact that μ is completely nonatomic [**3**, Theorems 1.17, 3.13] that there

Received by the editors November 4, 1982 and, in revised form, January 10, 1983 and June 7, 1983.

1980 *Mathematics Subject Classification*. Primary 46E30.

exists a sequence $\{f_n\}$ of simple functions such that $|f_n(x)| \leq |f(x)|$ a.e. for all n and $||f_n - f||_1 \to 0$. Choose δ which satisfies the property that whenever $0 < \mu(F) < \delta$,

$$\frac{1}{\mu(F)^{1/p}} \int_F |f| \, d\mu < \frac{\epsilon}{3}.$$

(This is possible since $f \in M_\phi^0$.) By Egoroff's theorem there exists a measurable set F with $\mu(F) < \delta$ and $f_n \to f$ uniformly on $X - F$. Choose n_0 such that

$$|f_{n_0}(x) - f(x)| < \frac{\epsilon}{3(\mu(X))^{1/q}} \quad \forall x \in X - F.$$

Now whenever $0 < \mu(E)$ we have

$$\begin{aligned}
\frac{1}{\mu(E)^{1/p}} \int_E |f_n - f| \, d\mu &= \frac{1}{\mu(E)^{1/p}} \int_{E \cap F} |f_n - f| \, d\mu + \frac{1}{\mu(E)^{1/p}} \int_{E-F} |f_n - f| \, d\mu \\
&\leq \frac{1}{\mu(E)^{1/p}} \left[\int_{E \cap F} |f_n| \, d\mu + \int_{E \cap F} |f| \, d\mu \right] + \frac{1}{\mu(E-F)^{1/p}} \int_{E-F} |f_n - f| \, d\mu \\
&\leq \frac{1}{(E \cap F)^{1-p}} \left[2 \int_{E \cap F} |f| \, d\mu \right] + \frac{\epsilon \mu(E-F)^{1/q}}{3(\mu(X))^{1/q}} \\
&\leq 2\epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Thus $||f_{n_0} - f||_{M_\phi} \leq \epsilon$ and this completes the proof.

THEOREM 1. *The dual space of M_ϕ^0 is isometrically isometric to N_ϕ.*

PROOF. We first prove that given a bounded linear positive functional T on M_ϕ^0, there exists a function $k \in N$ such that $Tf = \int_X kf \, d\mu$ and $||k||_{N_\phi} \leq ||T||$. If E is a μ-measurable subset of X, define $m(E) = T(\psi_E)$. Note that $m(E) \leq ||T|| \mu(E)^{1/q}$ since $||\psi_E||_{M_\phi} = \mu(E)^{1/q}$. It is easy to see that m is a measure which is absolutely continuous with respect to μ. If $k = dm/d\mu$ then $k \geq 0$, $k \in L_1(\mu)$ and $Tf = \int_X kf \, d\mu$ for all simple functions f. If $y \geq 0$, define $F_y = \{x, \, k(x) \geq y\}$. In particular, $F_n = \{x, \, k(x) \geq n\}$. Then $\{F_n\}$ is a decreasing sequence and if $E_n = F_n - F_{n+1}$, $\{E_n\}$ is a sequence of pairwise disjoint measurable sets. Let

$$f_n = \frac{\phi(\mu(F_0)) - \phi(\mu(F_1))}{\mu(E_0)} \psi_{E_0} + \frac{\phi(\mu(F_1)) - \phi(\mu(F_2))}{\mu(E_1)} \psi_{E_1} + \cdots + \frac{\phi(\mu(F_n))}{\mu(E_n)} \psi_{E_n}.$$

It is obvious that $f_n \in M_\phi^0$ and $||f_n||_{M_\phi} \leq 1$ [**2**, Theorem 5.5]. Hence $||T|| \geq \int_X kf_n \, d\mu$, but $\int_{E_n} k \, d\mu \geq n\mu(E_n)$. Hence

$$\begin{aligned}
||T|| \geq [\phi(\mu(F_1)) - \phi(\mu(F_2))] + [\phi(\mu(F_2)) - \phi(\mu(F_3))]2 \\
+ \cdots + [\phi(\mu(F_{n-1})) - \phi(u(F_n))](n-1) + n\phi(\mu(F_n))
\end{aligned}$$

i.e., $||T|| \geq \phi(\mu(F_1)) + \phi(\mu(F_2)) + \cdots + \phi(\mu(F_n))$ for all n. Thus $\sum_{k=1}^{\infty} \phi(\mu(F_k)) \leq ||T||$ and by the integral test $\int_1^\infty \phi(\mu(F_y)) \, dy \leq ||T||$, i.e. $k \in N_\phi$ and a standard approximation argument shows that $||k||_{N_\phi} \leq ||T||$.

On the other hand, for $k \in N_\phi$ and $f \in M_\phi$, we have $|\int kf \, d\mu| \leq ||k||_{N_\phi} ||f||_{M_\phi}$ [**2**, Theorem 4.4]. In particular, $||T|| \leq ||k||_{N_\phi}$. Thus there is an isometric isomorphism between the positive cones of $(M_\phi^0)^*$ and N_ϕ and the proof is complete.

References

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