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NONSEPARATING ALMOST CONTINUOUS RETRACTS OF I^n

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ABSTRACT. Compact almost continuous retracts of I^n ($n \geqslant 2$) do not separate E^n. Some other results that hold for continuous functions are also shown to hold for almost continuous functions. A result in [5] giving sufficient conditions for a set to be an almost continuous retract of I^n is examined further, and a method of constructing some almost continuous retracts of I^n is given.

1. Introduction. Almost continuous or connectivity functions sometimes obey the same type theorems that continuous functions obey. Sometimes they do not. For example, no connectivity function $f\colon S^n \to S^{n-1}$ commutes with the antipodal map [3], but some almost continuous function does [4]. Also, every connectivity or almost continuous retract of I^n has the fixed point property [2, 9].

Theorem 3.16 in [2] states that every connectivity retract of an n-cell in E^n ($n \geqslant 2$) is a nonseparating subcontinuum of E^n. Similarly, we can show that the closure of an almost continuous retract Y of an n-cell in E^n ($n \geqslant 2$) is a subcontinuum of E^n and that Y cannot separate E^n if Y is compact. Because it can apply to any compact almost continuous retract Y of I^n relative to I^{2n} which might not be an almost continuous retract relative to $I^n \times Y$, the preceding result is an extension of Theorem 4 in [5]. Some other results of algebraic topology are shown to hold for almost continuous functions. We also show that some of the sufficient conditions for a set M to be an almost continuous retract of I^n can be dropped from Theorem 1 of [5].

In the last section, some almost continuous retracts of I^n ($n \geqslant 2$) are constructed by removing infinitely many half-open n-cells from I^n.

2. Definitions and examples. A function $f\colon X \to Y$ is called *almost continuous relative to* $X \times Y$ if every neighborhood U of (the graph of) f contains some continuous function $g\colon X \to Y$. One familiar almost continuous function is any derivative $F'\colon [a, b] \to R$. We say that a closed subset K of $X \times Y$ is a *blocking set of* $X \times Y$ if K misses some function from X into Y but meets every continuous function from X into Y. A function $f\colon X \to Y$ is *almost continuous relative to* $X \times Z$ if $Y \subset Z$ and every neighborhood U of f in $X \times Z$ contains a continuous function $g\colon X \to Z$. It turns out that almost continuity of f relative to $X \times Y$ implies almost continuity of f relative to $X \times Z$, but not vice versa. A subcontinuum Y of X is an *almost continuous retract of* X *relative to* $X \times Z$ if $Y \subset Z$ and there is an almost continuous function $f\colon X \to Y$ relative to $X \times Z$ which is the identity on Y. If $X = Z = I^n$, we

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shall say simply that Y is an almost continuous retract of I^n or that f is an almost continuous retraction of I^n onto Y. When

$$Y = \left\{\left(x, \sin \frac{1}{x+1}\right): -1 < x \leqslant 1\right\} \cup \{(-1, 0)\}$$

and $I = [-1, 1]$, it is not hard to verify that vertical projection of I^2 onto Y is an example of an almost continuous retraction $f\colon I^2 \to Y$ relative to $I^2 \times Y$ with a nowhere dense graph in $I^2 \times Y$.

For a subset K of $I^n \times Y$, we let $p_x(K)$ denote the projection of K into the first coordinate space I^n.

3. Nonseparating property. We show how to modify the proof of Theorem 3.16 of **[2]** to obtain the following result.

THEOREM 1. *If Y is a compact almost continuous retract of I^n ($n \geqslant 2$), then Y does not separate E^n.*

PROOF. Let $f\colon I^n \to Y$ be an almost continuous retraction. If Y separates E^n, there exists an essential continuous function g of Y onto S^{n-1}, the boundary of the unit n-ball D^n in E^n. Since D^n is an AR, there is a continuous extension $G\colon I^n \to D^n$ of g. Of course, $G(I^n) \not\subset S^{n-1}$; however, $Gf\colon I^n \to D^n$ is almost continuous relative to $I^n \times D^n$ **[9]** and $Gf(I^n) \subset S^{n-1}$. Let B be the open n-ball in E^n with center at the origin O and radius 2, and let A be the open annulus $B - \{O\}$. Define

$$U = [(I^n - Y) \times A] \cup \{(p, q): p \in Y, q \in A, \text{and } d(q, g(p)) < 1\}$$

where d denotes the Euclidean metric on E^n. Since U is an open set containing Gf, there is a continuous function $h\colon I^n \to D^n$ such that $h \subset U$. Let ϕ be a retraction of A onto S^{n-1} such that $d(x, \phi(x)) < 1$ for all x in A. Then $\phi h(I^n) \subset S^{n-1}$ because $h(I^n) \subset A$. If $p \in Y$, then $d(g(p), h(p)) < 1$ and $d(h(p), \phi h(p)) < 1$. Therefore $d(g(p), \phi h(p)) < 2$. Since $g(p)$ and $\phi h(p)$ are not antipodal points of S^{n-1}, $\phi h \mid Y$ is homotopic to g. It follows that $\phi h\colon I^n \to S^{n-1}$ is essential, a contradiction.

COROLLARY 1. *There exists no almost continuous retraction of I^n onto its boundary.*

COROLLARY 2. *Let S^n be the boundary of the unit $(n+1)$-ball D^{n+1}. S^n is not contractible by means of an almost continuous homotopy relative to $S^n \times I \times D^{n+1}$.*

PROOF. Assume there is an almost continuous function $H\colon S^n \times I \to S^n$ relative to $S^n \times I \times D^{n+1}$ such that $H(x, -1) = x$ for all $x \in S^n$ and $H(S^n, 1) = \{q\}$. $S^n \times \{1\}$ bounds an $(n+1)$-cell $D_1 = D^{n+1} \times \{1\}$ such that $D_2 = (S^n \times I) \cup D_1$ is an $(n+1)$-cell. We can extend H to a function $F\colon D_2 \to S^n$ so that $F(D_1) = \{q\}$. If F were not almost continuous relative to $D_2 \times D^{n+1}$, then there would be a neighborhood U of F in $D_2 \times D^{n+1}$ that contains no continuous function from D_2 into D^{n+1}. Since $(D_2 \times D^{n+1}) - U$ misses $D_1 \times \{q\}$, there exists an open $(n+1)$-cell neighborhood V of q in D^{n+1} such that $D_1 \times V \subset U$. The neighborhood $U - [\mathrm{Bd}(D_1) \times (D^{n+1} - V)]$ of H contains a continuous function $h\colon S^n \times I \to S^n$. Because $h(\mathrm{Bd}\, D_1) \subset V$, h can be extended to a continuous function $g\colon D_2 \to D^{n+1}$ with $g \subset U$. On account of this contradiction, $F\colon D_2 \to S^n$ is almost continuous relative to $D_2 \times D^{n+1}$. But according to Corollary 1, this is impossible.

Other results that hold for continuous functions sometimes hold for almost continuous functions. Although we pointed out in the introduction that the Borsuk-Ulam Theorem does not hold for almost continuous functions [**4**], the theorem about the nonexistence of a continuous nonzero distribution f: $S^{2n} \to E^{2n}$ of tangent vectors on S^{2n} still does. For, if f: $S^{2n} \to E^{2n}$ is an almost continuous nonzero discontinuous distribution of tangent vectors, then pf: $S^{2n} \to S^{2n}$ is almost continuous where $p(x) = x/|x|$ [**9**]. Therefore, there is a point x_0 which pf either leaves fixed or maps to its antipode; otherwise, the neighborhood $S^{2n} \times S^{2n} - \{(x, y): y = \pm x\}$ of pf would contain a continuous function g: $S^{2n} \to S^{2n}$, an impossibility. Then x_0 is not orthogonal to $f(x_0)$.

EXAMPLE. Does Y have to be compact in Theorem 1? Of course, it is possible for $\bar{Y}$ to separate E^n if Y is a noncompact almost continuous retract of I^n; an example, M_3, was given in [**5**] of such a set Y. M_3 is a point on a circle along with a spiral beginning at the center and limiting on the circle. Theorem 1 of [**5**] gives sufficient conditions for a set M like M_3 to be an almost continuous retract of I^2 (and thus have the fixed point property); however, a gap in the proof of Theorem 1 occurs when it is stated that $p_x(K) - T$ is perfect, where $p_x(K)$ is the projection into I^n of a minimal blocking set K of $I^n \times M$ that misses some function f: $I^n \to M$ constructed there and T is the finite set of all isolated points of $p_x(K)$. For, M might not be compact and $p_x(K)$ might not be closed. The referee noticed that a gap seems to occur also in the proof of Lemma 2 in [**5**] when the dense arc component A_0 of M is apparently assumed to be open in M. (However, the spiral A_0 in the example M_3 mentioned above is open in M_3.) In correcting Theorem 1 of [**5**] with Theorem 2 below, we show we can omit from the hypothesis the conditions that M contains no simple closed curve and that the arc components of M except for A_0 are nowhere dense in M and finite in number.

The following lemma is a generalization of Theorem 2 in [**7**].

LEMMA 1. *If A_0 is an arcwise connected subset of E^n, then for each minimal blocking set K_0 of $I^n \times A_0$, $p_x(K_0)$ is a nondegenerate connected set.*

PROOF. It is obvious that $p_x(K_0)$ contains more than one point. Suppose $p_x(K_0)$ is not connected and $p_x(K_0) = A \cup B$ with A and B separated. Since I^n is completely normal, there exist in I^n disjoint open sets $U \supset A$ and $V \supset B$. No points of $p_x(K_0)$ are in $\mathrm{Bd}(U)$ or $\mathrm{Bd}(V)$. $K_0 \cap (U \times A_0)$ and $K_0 \cap (V \times A_0)$ are closed in $I^n \times A_0$, and neither can be a blocking set of $I^n \times A_0$. Therefore, there exist continuous functions g_1, g_2: $I^n \to A_0$ such that $p_x(K_0 \cap g_1) \subset V$ and $p_x(K_0 \cap g_2) \subset U$. For each positive integer i, let U_i be the open i^{-1}-neighborhood of $\mathrm{Bd}(U)$ in I^n, and let V_i be the open i^{-1}-neighborhood of $\mathrm{Bd}(V)$ in I^n. Let B_0 be an arc in A_0 that meets both $g_1(I^n)$ and $g_2(I^n)$, and let $C = g_1(I^n) \cup B_0 \cup g_2(I^n)$. It follows from the regular neighborhood collaring theorem [**8**, p. 36] that $g_i = g_1 | (\bar{U} - U_i) \cup g_2 | (\bar{V} - V_i)$ can be extended to a continuous function G_i: $I^n \to C$ so that $G_i(I^n - N_i)$ is a point for some regular neighborhood N_i in $U \cup V$ of a compact polyhedron containing $(\bar{U} - U_i) \cup (\bar{V} - V_i)$. For each i, G_i meets K_0 in some point of $(U_i \cup V_i) \times C$. Since K_0 is closed and C is compact, K_0 must meet $[\mathrm{Bd}(U) \cup \mathrm{Bd}(V)] \times C$, a contradiction.

THEOREM 2. *Let M be a subset of I^n with an arc component A_0 that is dense in M. Suppose there exists a function f_0: $\overline{M} \to M$ which is the identity on M and there exists a sequence of continuous functions $g_1, g_2, \ldots$ such that g_i: $I^n \to A_0$ and such that if P_1, $P_2, \ldots$ is a sequence of points of $\overline{M}$ converging to P, then $g_1(P_1), g_2(P_2), \ldots$ converges to $f_0(P)$. Then M is an almost continuous retract of I^n relative to $I^n \times M$.*

PROOF. We begin the proof as in [5]. Let θ be the set of all closed subsets L of $I^n \times M$ for which $p_x(L)$ has c-many points not in $\overline{M}$. Using transfinite induction, we can define a function f: $I^n \to M$ such that $f | \overline{M} = f_0$ and f meets each L in θ. We need only show that f is almost continuous relative to $I^n \times M$. Assume f is not. Then by Theorem 3 of [5], there exists a minimal blocking set K of $I^n \times M$ that misses f.

We show that for each x in I^n, $\{x\} \times A_0$ is not contained in $K \cap (\{x\} \times M)$. Assume that some $\{x\} \times A_0$ is contained in $K \cap (\{x\} \times M)$. Since $M \subset \overline{A}_0$ and K is closed in $I^n \times M$, then $\{x\} \times M = \{x\} \times (\overline{A}_0 \cap M) \subset K \cap (\{x\} \times M)$. Therefore, $(x, f(x)) \in K$ in contradiction to the fact that $K \cap f = \varnothing$. Then $\{x\} \times A_0 \not\subset K \cap (\{x\} \times M)$ for all $x \in I^n$.

For each $x \in I^n$, choose a point $(x, y) \in (\{x\} \times A_0) - (K \cap (\{x\} \times M))$. This defines a function h: $I^n \to A_0$ such that $y = h(x)$ and $K \cap (I^n \times A_0)$ misses h. If g: $I^n \to A_0$ is a continuous function, then g meets the blocking set K of $I^n \times M$ and hence meets $K \cap (I^n \times A_0)$. This shows that $K \cap (I^n \times A_0)$ is a blocking set of $I^n \times A_0$. According to Theorem 3 in [5], since h: $I^n \to A_0$ is not almost continuous and I^n is compact, there exists a minimal blocking set K_0 of $I^n \times A_0$ that misses h. Then K_0 meets each g_i in some point $(P_i, g_i(P_i))$.

Without loss of generality, we can suppose K_0 is contained in $K \cap (I^n \times A_0)$. For, in the proof of Theorem 3 in [5], one could have considered a chain of blocking sets contained in a prescribed blocking set (such as $K \cap (I^n \times A_0)$ here). Then Zorn's lemma would, as in [5], give a minimal blocking set now contained in the prescribed one.

By Lemma 1, $p_x(K_0)$ is a nondegenerate connected set. Then $p_x(K_0) \subset \overline{M}$, and so each P_i must lie in $\overline{M}$. Otherwise, if $p_x(K_0) \not\subset \overline{M}$, then $p_x(K_0)$ and hence $p_x(K)$ would have c-many points not in $\overline{M}$. This is impossible because f was constructed to meet each closed subset L of $I^n \times M$ whenever $p_x(L)$ has c-many points not in $\overline{M}$, yet $f \cap K = \varnothing$.

Continuing as in the proof in [5], we may assume that $P_1, P_2, \ldots$ converges to some point P. According to the hypothesis, $g_1(P_1), g_2(P_2), \ldots$ converges to $f_0(P) = f(P)$. Therefore $(P, f(P))$ is in the closed set K, a contradiction. This shows that f must be an almost continuous retraction of I^n onto M relative to $I^n \times M$.

4. Construction of some almost continuous retracts of I^n, $n \geqslant 2$. Kellum has noticed that, according to [6], if $M_1 \subset I^2$ is Knaster's indecomposable continuum with one endpoint, then M_1 is an almost continuous retract of I^2. In [1], Garrett shows that

$$M_2 = \mathrm{Cl}\left\{\left(x, \sin\frac{1}{x}\right): -1 \leqslant x \leqslant 1, x \neq 0\right\}$$

is an almost continuous retract of I^2. Both of these facts also come from the next general result because M_1 and M_2 can be obtained in the same way as the following continuum M. Here, interior and boundary of a cell are its combinatorial ones.

THEOREM 3. *Suppose $D_1, D_2, D_3, \ldots$ are topological n-cells in I^n with pairwise disjoint interiors such that each* Bd D_i *is the union of $(n-1)$-cells E_i and B_i with $B_i = \mathrm{Bd}(D_i) - \mathrm{Int}(E_i)$ and $E_i \subset \mathrm{Bd}\, I^n$. Let $M = I^n - \bigcup_{i=1}^{\infty}(D_i - B_i)$. Then M is an almost continuous retract of I^n.*

PROOF. Define $r: I^n \to M$ so that r is the identity on M and for $i = 1, 2, 3, \ldots, r|D_i$ is a (continuous) retraction of D_i onto B_i. Possibly, r might be discontinuous. We shall show r is an almost continuous retraction of I^n onto M.

Let $\varepsilon > 0$, and let U be the ε-neighborhood of r in I^{2n} and V be the $\varepsilon/2$-neighborhood of $r|M$ in U. It follows there is a neighborhood N of $r|M$ in V with the property that if $(z, y) \in N$, then $d(z, y) = \bar{d}((z, z), (z, y)) < \varepsilon/2$, where d and $\bar{d}$ denote the Euclidean metrics on E^n and E^{2n} respectively. There exists a positive integer m such that for all $i > m$, D_i is contained in the projection $p_x(N)$ of N into the first coordinate space I^n of $I^n \times I^n$. Define the continuous function $g: I^n \to I^n$ that is the identity on M and on D_i for all $i > m$ and so that $g = r$ on D_i for $i = 1, 2, \ldots, m$. Now suppose $i > m$ and $z \in D_i$. Since $z \in p_x(N)$, there is some y such that $(z, y) \in N$. Therefore $\bar{d}((z, z), (z, y)) < \varepsilon/2$, and there is some $w \in M$ such that $\bar{d}((z, y), (w, w)) < \varepsilon/2$. Then $\bar{d}((z, z), (w, w)) < \varepsilon$. That is, for every $i > m$, $g|D_i \subset U$. It follows that the neighborhood U of r contains the graph of the continuous function $g: I^n \to I^n$, and this shows r is almost continuous.

According to **[9]**, we have the following result.

COROLLARY 3. *Let M be the continuum in Theorem 3. Then M has the fixed point property.*

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