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A GENERALIZATION OF THE SIERPIŃSKI THEOREM

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ABSTRACT. Sierpiński's theorem admits the following generalization. Let n be a nonnegative integer and X a compact Hausdorff space. If $\{F_i \mid i \in \mathbf{N}\}$ is a countable closed covering of X such that $\dim(F_i \cap F_j) < n$ for distinct i and j in $\mathbf{N}$, then every continuous mapping from F_1 into the n-sphere S^n is extendable over X.

Consider the following well-known theorem (see Engelking [**3**, p. 440] or, for sets in $\mathbf{R}^n$, Sierpiński [**4**]).

THEOREM 1. *Let X be a continuum (i.e. a compact, connected Hausdorff space). If $\{F_i \mid i \in \mathbf{N}\}$ is a pairwise disjoint, closed covering of X then $F_i = X$ for some $i \in \mathbf{N}$.*

The aim of this paper is to prove the following assertion, where dim stands for the covering dimension.

THEOREM 2. *Let n be a nonnegative integer and X a compact Hausdorff space. If $\{F_i \mid i \in \mathbf{N}\}$ is a closed covering of X such that $\dim(F_i \cap F_j) < n$ for every i and j with $i \neq j$, then every continuous mapping from F_1 into the n-sphere S^n is continuously extendable over X.*

The reader is encouraged to verify that Theorem 1 follows easily from Theorem 2 if one substitutes $n = 0$. We first prove the theorem for metric spaces.

LEMMA 3. *Theorem 2 is valid in the class of metric spaces.*

PROOF. We shall work with the following induction hypothesis for $n = 0, 1, 2, \ldots$.

Let X be a compact metric space and M an absolute retract (AR). If $\{F_i \mid i \in \mathbf{N}\}$ is a closed covering of X such that $\dim(F_i \cap F_j) < n$ for every i and j with $i \neq j$, then every continuous $f: F_1 \to S^n \times M$ is extendable over X.

Consider the case $n = 0$, where we have S^n is the discrete doubleton $\{-1, 1\}$ and $\{F_i \mid i \in \mathbf{N}\}$ is pairwise disjoint. Assume that the closed set $A = f^{-1}(\{-1\} \times M) \subset F_1$ is nonempty. Let $\tilde{X}$ be the space we obtain from X by identifying A to a single point a, let $q: X \to \tilde{X}$ be the decomposition map and let C be the component of a in $\tilde{X}$. C is a continuum with the pairwise disjoint, closed covering $\{\{a\}, B \cap C\} \cup \{F_i \cap C \mid i \geq 2\}$, where $B = f^{-1}(\{1\} \times M)$. According to Theorem 1 we have $C = \{a\}$. Since $\tilde{X}$ is a compact Hausdorff space there is a clopen subset O of $\tilde{X}$ with $a \in O$ and $O \cap B = \emptyset$. Because M is an AR we can find

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continuous mappings $g_1: q^{-1}(O) \to \{-1\} \times M$ and $g_2: X \backslash q^{-1}(O) \to \{1\} \times M$ such that $g_1|A = f|A$ and $g_2|B = f|B$. Then $g_1 \cup g_2: X \to S^0 \times M$ is as required.

Assume now that the induction hypothesis holds for n. Let $\{F_i \mid i \in \mathbf{N}\}$ be a closed covering of X such that $\dim(F_i \cap F_j) \leq n$ for $i \neq j$, and let $f: F_1 \to S^{n+1} \times M$ be continuous. According to the countable sum theorem (Engelking [**2**, 3.1.8]) the set $B = \bigcup\{F_i \cap F_j \mid i,\ j \in \mathbf{N} \text{ with } i \neq j\}$ has covering dimension $\leq n$. Let x_1 and x_2 be two distinct points in S^{n+1} and note that $S^{n+1} \backslash \{x_1, x_2\}$ is homeomorphic to $S^n \times \mathbf{R}$. Using the separation theorem (Engelking [**2**, 4.1.13]) we find a closed covering $\{H_1, H_2\}$ of X such that $H_j \cap f^{-1}(\{x_j\} \times M) = \emptyset$ for $j = 1, 2$, and $\dim(H_1 \cap H_2 \cap B) < n$.

Consider the compact space $X' = H_1 \cap H_2$ and its closed covering $\{F_i \cap X' \mid i \in \mathbf{N}\}$. Obviously, we have

$$\dim(F_i \cap F_j \cap X') \leq \dim(B \cap X') < n \quad \text{for } i \neq j.$$

Observe that $f|F_1 \cap X'$ is a continuous mapping into $(S^{n+1} \backslash \{x_1, x_2\}) \times M$, which space is homeomorphic to $S^n \times \mathbf{R} \times M$. Since $\mathbf{R} \times M$ is, as a product of AR's, itself an AR we may apply the induction hypothesis to find a continuous

$$g: X' \to (S^{n+1} \backslash \{x_1, x_2\}) \times M$$

with $g|F_1 \cap X' = f|F_1 \cap X'$. Observing that $S^{n+1} \backslash \{x_j\}$ is homeomorphic to $\mathbf{R}^{n+1}$ select continuous functions

$$h_j: H_j \to (S^{n+1} \backslash \{x_j\}) \times M, \qquad j = 1, 2,$$

with $h_j|X' = g$ and $h_j|H_j \cap F_1 = f|H_j \cap F_1$. Then $h = h_1 \cup h_2$ is a continuous mapping from X into $S^{n+1} \times M$ which extends f and the lemma is proved.

NOTATIONS. Let d be a pseudo-metric on a space X. For $\varepsilon > 0$ and $x \in X$ let $S_\varepsilon^d(x) = \{y \in X \mid d(y, x) < \varepsilon\}$ be the ε-ball with respect to d. $\mathcal{T}_d$ denotes that topology for which $\{S_\varepsilon^d(x) \mid x \in X \text{ and } \varepsilon > 0\}$ is an open basis. d is called an *admissible* metric on X if $\mathcal{T}_d$ is contained in the topology of X. Let $\mathcal{U} = \{U_i \mid i = 1, \ldots, k\}$ be a finite collection of subsets of X. $\operatorname{mesh}_d \mathcal{U}$ is the maximum of the d-diameters of the elements of $\mathcal{U}$. If every point of X is contained in at most n (≥ 0) members of $\mathcal{U}$ then we denote this by $\operatorname{ord} \mathcal{U} \leq n$. $\mathcal{V} = \{V_i \mid i = 1, \ldots, k\}$ is called a *shrinking* of $\mathcal{U}$ if for every $i \leq k$, $V_i \subset U_i$.

LEMMA 4. *Let n be a nonnegative integer. Suppose that X is a normal space and F a closed subset with* $\dim(F) < n$. *If $\mathcal{U} = \{U_i \mid i = 1, \ldots, k\}$ is a finite open covering of F in X then there is an admissible pseudo-metric d on X and a shrinking $\mathcal{V} = \{V_i \mid i = 1, \ldots, k\}$ of $\mathcal{U}$ such that $\mathcal{V} \subset \mathcal{T}_d$,* $\operatorname{ord} \mathcal{V} \leq n$ *and $F \subset \bigcup \mathcal{V}$.*

PROOF. Let $\mathcal{U} = \{U_i \mid i = 1, \ldots, k\}$ be an open covering of F in X. Consider $\mathcal{U}_F = \{U_i \cap F \mid i = 1, \ldots, k\}$ and select an F-open shrinking $\mathcal{V} = \{V_i \mid i = 1, \ldots, k\}$ of $\mathcal{U}_F$ with $\bigcup \mathcal{V} = F$ and $\operatorname{ord} \mathcal{V} \leq n$. Since F is a normal space we may assume that the V_i's are cozero-sets in F such that the closure of V_i is contained in U_i. The set F is closed in the normal space and hence there are cozero-sets $\hat{V}_i$ in X such that $\hat{V}_i \cap F = V_i$. Moreover, since $\operatorname{Cl}(V_i) \subset U_i$ we may assume that $\hat{V}_i \subset U_i$. Using real valued mappings that correspond with the cozero-sets $\hat{V}_1, \ldots, \hat{V}_k$ we construct an admissible pseudo-metric d on X such that $\hat{V}_i \in \mathcal{T}_d$ for $i \leq k$. Define now for every $U \in \{O \cap F \mid O \in \mathcal{T}_d\}$ the set

$$U' = \{x \in X \mid d(x, U) < d(x, F \backslash U)\} \in \mathcal{T}_d,$$

where $d(x,\emptyset)=\infty$. It is easily verified that $U'\cap F=U$ and $U'\cap V'=(U\cap V)'$. This implies that $\operatorname{ord}\{(V_i)' \mid i=1,\dots,k\}\le n$ and hence that $\{\hat{V}_i\cap(V_i)' \mid i=1,\dots,k\}$ meets the requirements of this lemma.

We have now concluded the preliminaries and shall prove the theorem.

PROOF OF THEOREM 2. Let f be a continuous function from F_1 into S^n. We construct (inductively) a sequence $\rho_1,\rho_2,\rho_3,\dots$ of admissible pseudo-metrics on X. Let S^n be represented by the boundary ∂I^{n+1} of the cube I^{n+1} and let $\overline{f}\colon X\to I^{n+1}$ be a continuous extension of f. Take for ρ_1 the preimage under $\overline{f}$ of a metric that corresponds with the euclidean topology on I^{n+1} and that is bounded by 1. Suppose that we have constructed ρ_m. Let $\alpha=(i,j)\in A=\{(k,l)\mid k,\ l\in\mathbf{N}\text{ with }k\ne l\}$. With Lemma 4 we can find an admissible pseudo-metric d_α on X and a finite covering $\mathcal{V}$ of $F_i\cap F_j$ such that $\mathcal{V}\subset\mathcal{T}_{d_\alpha}$, $\operatorname{ord}\mathcal{V}\le n$ and $\operatorname{mesh}_{\rho_m}\mathcal{V}<1/m$. Let $O=\bigcup\mathcal{V}$ and consider the disjoint closed sets $F_i\backslash O$ and $F_j\backslash O$. Since X is normal we may assume that $d_\alpha(F_i\backslash O,\ F_j\backslash O)>0$. Moreover, let d_α be bounded by 1. Define now the admissible pseudo-metric ρ_{m+1} on X by

$$\rho_{m+1}(x,y)=\max\{\rho_m(x,y)\}\cup\left\{\frac{1}{i\cdot j}d_\alpha(x,y)\mid\alpha=(i,j)\in A\right\}.$$

If we put

$$\rho(x,y)=\max\{\rho_m(x,y)/m\mid m\in\mathbf{N}\}\qquad\text{for } x,\ y\in X,$$

then it is obvious that

$$\mathcal{T}_{\rho_1}\subset\mathcal{T}_{\rho_2}\subset\cdots\subset\mathcal{T}_{\rho_m}\subset\cdots\subset\mathcal{T}_\rho\subset\mathcal{O},$$

where $\mathcal{O}$ is the collection of open subsets of X.

Define the equivalence relation $\sim$ on X by $x\sim y\Leftrightarrow\rho(x,y)=0$. Let $\tilde{X}$ be the quotient space $X/\!\sim$ and let $q\colon X\to\tilde{X}$ be the decomposition mapping. Make a metric space of $\tilde{X}$ by putting $\tilde{\rho}(q(x),q(y))=\rho(x,y)$ for $x,\ y\in X$. Since X is compact and ρ is admissible, q is a closed mapping and $(X,\tilde{\rho})$ is a compact metric space. If $x,\ y\in F_1$ such that $q(x)=q(y)$ then $\rho(x,y)=0$ and hence $\rho_1(x,y)=0$. Consequently, $f(x)=f(y)$ and since q is closed there is a continuous $\tilde{f}\colon q(F_1)\to S^n$ with $\tilde{f}\circ q\mid F_1=f$. Obviously, $\{q(F_i)\mid i\in\mathbf{N}\}$ is a closed covering of $\tilde{X}$.

It remains to show that $\dim(q(F_i)\cap(F_i))<n$ for $i\ne j$. Let $\mathcal{U}$ be a finite open covering of $q(F_i)\cap q(F_j)$ in $\tilde{X}$. We have that

$$F_i\cap F_j\subset\bigcup q^{-1}[\mathcal{U}]=\bigcup\{q^{-1}(U)\mid U\in\mathcal{U}\}.$$

Since $F_i\cap F_j$ is compact and since $q^{-1}[\mathcal{U}]\subset\mathcal{T}_\rho$ there is a Lebesgue number $1/m$ for $q^{-1}[\mathcal{U}]$, i.e. for every $x\in F_i\cap F_j$, $S^\rho_{1/m}(x)$ is contained in some element of $q^{-1}[\mathcal{U}]$. Since $\rho_1\le\rho_2\le\rho_3\le\cdots$ and every ρ_k is bounded by 1 we have that

$$S^{\rho_m}_{1/m}(x)\subset S^\rho_{1/m}(x)\quad\text{for every } x\in X.$$

Now there is a pseudo-metric d on X and a finite subcollection $\mathcal{V}$ of $\mathcal{T}_d$ such that $d/(i\cdot j)\le\rho_{m+1}$, $\operatorname{mesh}_{\rho_m}\mathcal{V}<1/m$, $\operatorname{ord}\mathcal{V}\le n$ and $d(F_i\backslash O,F_j\backslash O)>0$, where $O=\bigcup\mathcal{V}$. Every element V of $\mathcal{V}$ is in $\mathcal{T}_\rho$ and hence $q^{-1}(q(V))=V$. This implies that $q[\mathcal{V}]=\{q(V)\mid V\in\mathcal{V}\}$ is an open collection in $\tilde{X}$ and has the property $\operatorname{ord}q[\mathcal{V}]\le\operatorname{ord}\mathcal{V}\le n$. We may assume that every member of $\mathcal{V}$ has points in common with $F_i\cap F_j$. It follows that for every $V\in\mathcal{V}$ there is a $U\in\mathcal{U}$ and an

$x \in F_i \cap F_j$ such that $V \subset S^{\rho_m}_{1/m}(x) \subset q^{-1}(U)$ and hence $q(V) \subset U$. In order to prove that $q(F_i) \cap q(F_j) \subset \bigcup \mathcal{V}$ select a $y \in F_i$ and a $y' \in F_j$ such that $q(y) = q(y')$. Suppose that $y \notin O$. Since $O \in \mathcal{T}_\rho$, $q^{-1}(q(O)) = O$ and hence $y' \notin O$. Then

$$\begin{aligned}\rho(y, y') &\geq \frac{1}{m+1}\rho_{m+1}(y, y') \geq \frac{1}{(m+1)\cdot i \cdot j} d(y, y') \\ &\geq \frac{1}{(m+1)\cdot i \cdot j} d(F_i \backslash O, F_j \backslash O) > 0\end{aligned}$$

which contradicts $q(y) = q(y')$. This implies that $y \in O$ and $q(y) \in \bigcup q[\mathcal{V}]$. So we have proved that every finite open covering of $q(F_i) \cap q(F_j)$ can be refined by a finite open covering of order $\leq n$. This means that $\dim(q(F_i) \cap q(F_j)) < n$.

We have arrived at a position where we can apply Lemma 3, yielding the existence of a continuous $g\colon \tilde{X} \to S^n$ that extends $\tilde{f}$. Then $g \circ q$ is the required extension of f.

REMARKS. We mention two applications of the theorem. The space $\mathbf{R}^n$ cannot be written as union of a sequence $F_1, F_2, F_3, \ldots$ of compact sets with $\dim(F_i \cap F_j) < n - 1$ for $i \neq j$. This follows directly from the fact that the identity mapping $\partial I^n \to \partial I^n$ is not extendable over I^n.

The second application concerns boundary sets in the Hilbert cube Q. If $(F_i)_{i=1}^\infty$ is a sequence of closed subsets of Q such that $Q \backslash \bigcup_{i=1}^\infty F_i$ is homeomorphic to the Hilbert space l^2, then there is for each $n \in \mathbf{N}$ an infinite set $\{i_m \mid m \in \mathbf{N}\} \subset \mathbf{N}$ such that $\dim(F_{i_m} \cap F_{i_{m+1}}) > n$ for every $m \in \mathbf{N}$. This result was established in the author's thesis [**1**, 5.4.6], where the theorem plays a key role.

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