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TWO PROBLEMS OF DOWKER

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ABSTRACT. An old question first raised by Hugh Dowker is asked again and two related problems are solved.

Question 1. Does there exist a set X and a filter F on X having properties (1.a) and (1.b)?

(1.a) If $f: X \to F$, then there are $x \neq y$ in X with $x \in f(y)$ and $y \in f(x)$.

(1.b) If $Y \subset X$, then there is $f: X \to F$ such that, if $y \in Y$ and $x \in X - Y$, either $x \notin f(y)$ or $y \notin f(x)$.

Question 2. Does there exist a set X and, for each $x \in X$, a set F_x of subsets of X closed under finite intersections and having properties (2.a) and (2.b)?

(2.a) If $f(x) \in F_x$ for all $x \in X$, then there are $x \neq y$ in X with $x \in f(y)$ and $y \in f(x)$.

(2.b) If $Y \subset X$, then there are $f(x) \in F_x$ for all $x \in X$, such that, for all $y \in Y$ and $x \in X - Y$, either $x \notin f(y)$ or $y \notin f(x)$.

Question 3. Does there exist a T_4, not paracompact, simplicial complex K, with the star of each vertex open? (We assume that the dimension of K is 1 and that, if $x \neq y$ are vertices, there is at most one 1-simplex $[x, y]$ of K having x and y as its end points. Observe that the vertices of K testify that K is not collectionwise Hausdorff; that is they cannot be covered by a family of disjoint open sets each containing only one vertex.)

A *yes* answer to Question 1 implies a *yes* answer to Question 2 which implies a *yes* answer to Question 3 (see Lemma 1). While trying to anwer the topological Question 3, Hugh Dowker was led to ask Question 1, the title question of [**1**].

So far as I know, Question 1 is still unanswered. The purpose of this note is to prove that the answer to Question 2 (and thus also Question 3) is *yes*. However, I conjecture that the answer to Question 1 is *no*.

LEMMA 1. *If the answer to Question* 2 *is yes, so is the answer to Question* 3.

PROOF. Suppose X and $\{F_x \mid x \in X\}$ give a *yes* answer to Question 2.

We can assume that for each $x \in X$, $\bigcap F_x = \varnothing$. For otherwise, define $F'_x = \{f - \bigcap F_x \mid f \in F_x\}$. Clearly $\bigcap F'_x = \varnothing$ and $\{F'_x \mid x \in X\}$ has all the properties desired of $\{F_x \mid x \in X\}$. To see that (2.a) holds, assume on the contrary that for every $x \in X$, there is $f(x) \in F_x$ such that $x \neq y$ implies that either $y \notin (f(x) - \bigcap F_x)$ or $x \notin (f(y) - \bigcap F_y)$.

By (2.b) for all $x \in X$, there is $g_x(y) \in F_y$ for all $y \in X$ such that, if $y \neq x$ in X, then either $x \notin g_x(y)$ or $y \notin g_x(x)$. For each $x \in X$ suppose $f(x) \in F_x$; define $k(x) = f(x) \cap g_x(x)$. By (2.a) there are $x \neq y$ in X such that $x \in k(y)$ and

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$y \in k(x)$. Since $y \in k(x) \subset g_x(x)$, $x \notin g_x(y)$. But $x \in k(y) \subset f(y)$ so, since $g_x(y) \in F_y$, $x \in (f(y) - g_x(y)) \subset (f(y) - \bigcap F_y)$. Similarly $y \in (f(x) - \bigcap F_x)$. Thus (2.a) holds for $\{F'_x \mid x \in X\}$, and we can assume that $\bigcap F_x = \varnothing$ for all $x \in X$.

If $x \neq y$ in X, choose a unit interval $[x, y] = [y, x]$ between x and y and let $d_{[x,y]}$ be the distance function on this interval. We let K be the complex $X \cup \{[x, y] \mid x \neq y \text{ in } X\}$; it remains to give the topology of $\bigcup K$ at the vertices. For $x \in X$, $A \in F_x$, and $0 < \epsilon < \frac{1}{4}$, let

$$U_{A,\epsilon}(x) = \{x\} \cup \{p \in [x, y] \mid x \neq y,\ y \in X - A, \text{ and } d_{[x,y]}(x, p) < \epsilon\}$$
$$\cup \{p \in [x, y] \mid x \neq y,\ y \in A, \text{ and } d_{[x,y]}(x, p) < \{\tfrac{1}{2}\} + \epsilon\}.$$

Let $\{U_{A,\epsilon}(x) \mid A \in F_x \text{ and } 0 < \epsilon < \frac{1}{4}\}$ be a basis for the topology of K at x. This complex trivially has the desired properties.

THEOREM. *The answer to Question* 2 *is yes.*

PROOF. Let X be the set of all ordinals less than $c = |\mathbf{R}|$. Let $C = \{A \subset X \mid A \text{ is countable}\}$ and $\mathcal{B} = \{\mathcal{A} \subset C \mid \mathcal{A} \text{ is finite}\}$. Index $\{k\colon \mathbf{N}^2 \to (X \times \mathcal{B})\}$ as $\{k_x \mid x < c\}$ in such a way that each k is k_x for c many x's.

For all $y \in X$ we define

$$A_y = \bigcup\{\bigcup \mathcal{A} \mid \mathcal{A} \text{ is the second term of a member of the range of } k_y\}.$$

If $x \in X$ and $Y \subset X$, we define $g_x(Y) = \{y \in X \mid x < y \text{ and there is } \langle x, \mathcal{A}\rangle \in (\text{range } k_y) \text{ with } (Y \cap A_y) \in \mathcal{A}\}$. Now define

$$f_x(Y) = \left\{ y \in X \;\middle|\; \begin{array}{l} \text{(a) } x \in Y \text{ if and only if } y \in Y \text{ or} \\ \text{(b) } x < y \text{ and } y \in g_x(Y) \text{ or} \\ \text{(c) } y < x \text{ and } x \notin g_y(Y) \end{array} \right\}.$$

For $x \in X$, define $F_x = \{\bigcap_{Y \in \mathcal{Y}} f_x(Y) \mid \mathcal{Y} \text{ is a finite set of subsets of } X\}$. Clearly $\{F_x \mid x \in X\}$ satisfies (2.b): simply define $f(x) = f_x(Y)$. It remains to show that it satisfies (2.a).

Suppose that for every $x \in X$, $f_x \in F_x$. We find $x \neq y$ in X with $x \in f_y$ and $y \in f_x$. Let $\mathcal{Y}_x$ denote the finite set of subsets of X such that $f_x = \bigcap\{f_x(Y) \mid Y \in \mathcal{Y}_x\}$.

We first choose a function $h\colon \mathbf{N}^2 \to X$ by induction. For $i \in \mathbf{N}$ we let $H_i = \{h(i', j) \mid i' < i\}$ and we assume that H_i is finite. Every $y \in X$ has an "ith characteristic function" $c_y\colon (\bigcup_{x \in H_i} \mathcal{Y}_x) \to 2$ defined by $c_y(Y) = 1$ if and only if $y \in Y$. If $c_1, \ldots, c_n$ is a listing of the possible ith characteristic functions for terms of $X - H_i$, for each $j \leq n$ choose $y \in X - H_i$ with $c_y = c_j$; then define $h(i, j) = y$. For $j > n$, define $h(i, j) = h(i, 1)$.

Let $H =$range of h and $\mathcal{Y} = \bigcup\{\mathcal{Y}_x \mid x \in H\}$. If $x \in H$, $x \in H_i$ for a minimal i; define $\mathcal{Y}^*_x = \mathcal{Y}_x - \bigcup\{\mathcal{Y}_z \mid z \in H_{i-1}\}$. For $Y \in \mathcal{Y}$ there is a unique $i = i_Y \in \mathbf{N}$ such that $Y \in \mathcal{Y}^*_x$ for some $x \in H_i$.

If $Y \neq Z$ in $\mathcal{Y}$ choose $a_{Y,Z}$ in either $Y - Z$ or $Z - Y$; then let $A = \{a_{Y,Z} \mid Y \neq Z \text{ in } \mathcal{Y}\}$. For $Y \in \mathcal{Y}$, let $A_Y = A \cap Y$. Observe that A is countable, $A = \bigcup\{A_Y \mid Y \in \mathcal{Y}\}$, and that if $Y \neq Z$ in $\mathcal{Y}$, then $A_Y \neq A_Z$. For each $x \in H$, let $\mathcal{A}_x = \{A_Y \mid Y \in \mathcal{Y}^*_x\}$, and define $k\colon \mathbf{N}^2 \to (X \times \mathcal{B})$ by $k(i, j) = \langle x, \mathcal{A}_x\rangle$ if $h(i, j) = x$.

Choose $y \in X$ such that $y > x$ for all $x \in H$ and $k_y = k$. Since $\mathcal{Y}_y$ is finite, so is $\mathcal{Y}^* = \{Y \in \mathcal{Y} \mid A_Y = (A \cap Z) \text{ for some } Z \in \mathcal{Y}_y\}$. Thus there is an $i \in \mathbf{N}$ such that $i > i_Y$ for all $Y \in \mathcal{Y}^*$. Since $y \in X - H$, there is $x \in H_{i+1} - H_i$ such that the ith characteristic functions of x and y are the same.

We show that $x \in f_y$ and $y \in f_x$, thus proving our theorem.

Recall that $f_x = \bigcap\{f_x(Y) \mid Y \in \mathcal{Y}_x\}$. If $Y \in (\mathcal{Y}_x - \mathcal{Y}_x^*)$, then $i_Y \leq i$ and, since x and y have the same ith characteristic function, $x \in Y$ if and only $y \in Y$. Thus, by (a), $y \in f_x(Y)$. If $Y \in \mathcal{Y}_x^*$, then since $k_y = k$, $\langle x, \mathcal{A}_x\rangle \in (\text{range } k_y)$. Since $A_Y = A$, $A_Y = (Y \cap A) \in \mathcal{A}_x$. By (b), since $x < y$ and $y \in g_x(Y)$, $y \in f_x(Y)$. So $y \in f_x$.

To show that $x \in f_y$, suppose that $Y \in \mathcal{Y}_y$. By our choice of i, if $A \cap Y = A_Z$ for some $Z \in \mathcal{Y}$, then $i_Z < i$; so $A_y \cap Y = A \cap Y \notin \mathcal{A}_x$. Thus $y \notin g_x(Y)$ since $\langle x, \mathcal{A}_x\rangle$ is the only point in the range of k having x as its first coordinate. So, by (c), $x \in f_y(Y)$ and $x \in f_y$ and our proof is complete.

LEMMA 2. *The existence of a first countable, T_4, noncollectionwise Hausdorff space implies the answer to Question* 2 *is yes with the F_x's each being countable.*

PROOF. Let S be the space and X the closed discrete set of points in S which cannot be separated; we assume without loss of generality, that the points of $X - S$ are isolated. For each $x \in X$, choose an open basis $U_1(x) \supset U_2(x) \supset \cdots$ for the topology at x with $\overline{U_1(x)} \cap X = \{x\}$. For each $x \in X$, let $f_n(x) = \{y \in X - \{x\} \mid U_n(x) \cap U_n(y) \neq \varnothing\}$. Then, if $F_x = \{f_n(x) \mid n \in \mathbf{N}\}$, $\{F_x \mid x \in X\}$ has the desired properties.

Comments on cardinalities. It has long been known that there is a model for ZFC in which there is a T_4, noncollectionwise Hausdorff space of countable character. In another model, all T_4, noncollectionwise Hausdorff spaces have character greater than c.

If X and $\{F_x \mid x \in X\}$ yield a *yes* answer to Question 2, the manifold K constructed as in the proof of Lemma 1 is a T_4, noncollectionwise Hausdorff space and each vertex x has character $\leq |F_x|$ (or countable if F_x is countable). By the proofs of Lemmas 1 and 2, the existence of X and $\{F_x \mid x \in X\}$ yielding a *yes* answer to Question 2 with each F_x countable is equivalent to the existence of a K yielding a *yes* answer to Question 3 of countable character is equivalent to the existence of any T_4, noncollectionwise Hausdorff space of countable character; and all are undecidable in ZFC. Similarly the existence of X and $\{F_x \mid x \in X\}$ yielding a *yes* answer to Question 2 with each F_x of cardinality $\leq c$ or a K yielding a *yes* answer to Question 3 of character $\leq c$ is undecidable in ZFC.

The X of our Theorem has cardinality c. The cardinality of a particular F_x depends on the choice of $\{k_y \mid y \in X\}$, but it could be as much as 2^c. There is a consistency example with the cardinality of $X = \omega_1$ and all F_x are countable. But for a real example, one cannot hope to do much better. However, none of these solutions of Question 2 are apt to lead to a solution of Question 1 since Dowker proves that any X and F yielding a *yes* answer to Question 1 must have the cardinality of X greater than ω_2. For completeness I give some of Dowker's results.

Comments on Question 1 (*Dowker*). Suppose that X and F yield a *yes* answer to Question 1 and that the cardinality α of X is minimal for such to exist. Then $Y \in F$ implies Y has cardinality α since $\{A \cap Y \mid A \in F\}$ would be an "F" set for Y. Let β be the minimal cardinality of a subset of X whose complement is *not* in F. By (1.a), no nested sequence of members of F has empty intersection. If $Y = \{y_\gamma \mid \gamma < \beta\} \subset X$ and for each $\delta < \beta$, $Y_\delta = X - \{y_\gamma \mid \gamma < \delta\}$, then $\{Y_\delta \mid \delta < \beta\}$ is a nested sequence of members of F; hence $\beta < \alpha$. Since there is such a Y with $(X - Y) \notin F$, and $\beta < \alpha$ implies $Y \notin F$, F is *not* an ultrafilter. Assuming $(X - Y) \notin$

F, let $f\colon X \to F$ satisfy (1.b) and, for $\delta < \beta$, define $Y^*_\delta = Y_\delta \cap f(Y_\delta)$. Then $Y^*_\delta \in F$ and $\bigcap_{\delta<\beta} Y^*_\delta = \varnothing$. It is impossible to have $\beta \leq \omega$ since then $Z_\delta = \bigcap_{\gamma\leq\delta} Y^*_\gamma \in F$ and $\{Z_\delta \mid \delta < \beta\}$ is a nested sequence with empty intersection. Hence $\omega < \beta < \alpha$ and $\alpha \geq \omega_2$.

Bibliography

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