TWO PROBLEMS OF DOWKER

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Abstract. An old question first raised by Hugh Dowker is asked again and two related problems are solved.

Question 1. Does there exist a set X and a filter F on X having properties (1.a) and (1.b)?

(1.a) If \( f : X \to F \), then there are \( x \neq y \) in X with \( x \in f(y) \) and \( y \in f(x) \).
(1.b) If \( Y \subset X \), then there is \( f : X \to F \) such that, if \( y \in Y \) and \( x \in X - Y \), either \( x \notin f(y) \) or \( y \notin f(x) \).

Question 2. Does there exist a set X and, for each \( x \in X \), a set \( F_x \) of subsets of X closed under finite intersections and having properties (2.a) and (2.b)?

(2.a) If \( f(x) \in F_x \) for all \( x \in X \), then there are \( x \neq y \) in X with \( x \in f(y) \) and \( y \in f(x) \).
(2.b) If \( Y \subset X \), then there are \( f(x) \in F_x \) for all \( x \in X \), such that, for all \( y \in Y \) and \( x \in X - Y \), either \( x \notin f(y) \) or \( y \notin f(x) \).

Question 3. Does there exist a \( T_4 \), not paracompact, simplicial complex \( K \), with the star of each vertex open? (We assume that the dimension of \( K \) is 1 and that, if \( x \neq y \) are vertices, there is at most one 1-simplex \([x, y]\) of \( K \) having \( x \) and \( y \) as its end points. Observe that the vertices of \( K \) testify that \( K \) is not collectionwise Hausdorff; that is they cannot be covered by a family of disjoint open sets each containing only one vertex.)

A yes answer to Question 1 implies a yes answer to Question 2 which implies a yes answer to Question 3 (see Lemma 1). While trying to answer the topological Question 3, Hugh Dowker was led to ask Question 1, the title question of [1].

So far as I know, Question 1 is still unanswered. The purpose of this note is to prove that the answer to Question 2 (and thus also Question 3) is yes. However, I conjecture that the answer to Question 1 is no.

Lemma 1. If the answer to Question 2 is yes, so is the answer to Question 3.

Proof. Suppose \( X \) and \( \{F_x \mid x \in X\} \) give a yes answer to Question 2.

We can assume that for each \( x \in X \), \( \bigcap F_x = \emptyset \). For otherwise, define \( F'_x = \{ f - \bigcap F_x \mid f \in F_x \} \). Clearly \( \bigcap F'_x = \emptyset \) and \( \{F'_x \mid x \in X\} \) has all the properties desired of \( \{F_x \mid x \in X\} \). To see that (2.a) holds, assume on the contrary that for every \( x \in X \), there is \( f(x) \in F_x \) such that \( x \neq y \) implies that either \( y \notin (f(x) - \bigcap F_x) \) or \( x \notin (f(y) - \bigcap F_y) \).

By (2.b) for all \( x \in X \), there is \( g_x(y) \in F_y \) for all \( y \in X \) such that, if \( y \neq x \) in \( X \), then either \( x \notin g_x(y) \) or \( y \notin g_x(x) \). For each \( x \in X \) suppose \( f(x) \in F_x \); define \( k(x) = f(x) \cap g_x(x) \). By (2.a) there are \( x \neq y \) in \( X \) such that \( x \in k(y) \) and...
y \in k(x). Since y \in k(x) \subset g_x(x), x \notin g_x(y). But x \in k(y) \subset f(y) so, since g_x(y) \in F_y,
x \in (f(y) - g_x(y)) \subset (f(y) \cap F_x). Similarly y \in (f(x) \cap F_x). Thus (2.a) holds for
\{F'_x | x \in X\}, and we can assume that ∩F_x = \emptyset for all x \in X.

If x \neq y in X, choose a unit interval [x, y] = [y, x] between x and y and let d_{[x, y]}
be the distance function on this interval. We let K be the complex X \cup \{[x, y] | x \neq
y in X\}; it remains to give the topology of ∪K at the vertices. For x \in X, A \in F_x, and 0 < \epsilon < \frac{1}{4}, let

\begin{align*}
U_{A,\epsilon}(x) &= \{x\} \cup \{p \in [x, y] | x \neq y, y \in X - A, \text{ and } d_{[x, y]}(x, p) < \epsilon\} \\
&\quad \cup \{p \in [x, y] | x \neq y, y \in A, \text{ and } d_{[x, y]}(x, p) < \{\frac{1}{2} + \epsilon\}\}.
\end{align*}

Let \{U_{A,\epsilon}(x) | A \in F_x \text{ and } 0 < \epsilon < \frac{1}{4}\} be a basis for the topology of K at x. This
complex trivially has the desired properties.

**Theorem.** The answer to Question 2 is yes.

**Proof.** Let X be the set of all ordinals less than \(c = |\mathbb{R}|\). Let C = \{A \subset X | A is countable\} and B = \{A \subset C | A is finite\}. Index \(\{k: \mathbb{N}^2 \to (X \times \mathcal{B})\}\) as \(\{k_x | x < c\}\) in such a way that each k is k_x for c many x’s.

For all y \in X we define

\[ A_y = \bigcup \{\bigcup A | A \text{ is the second term of a member of the range of } k_y\}. \]

If x \in X and Y \subset X, we define \(g_x(Y) = \{y \in X | x < y \text{ and there is } (x, A) \in \text{(range } k_y) \text{ with } (Y \cap A)^x \in A\}. \) Now define

\[ f_x(Y) = \begin{cases} 
(x) \in Y \text{ if and only if } y \in Y \text{ or} \\
(b) y < x \text{ and } y \in g_x(Y) \text{ or} \\
(c) x < y \text{ and } x \notin g_y(Y) 
\end{cases}. \]

For x \in X, define \(F_x = \{\bigcap_{Y \in y} f_x(Y) | y \text{ is a finite set of subsets of } X\}. \) Clearly \(\{F_x | x \in X\}\) satisfies (2.b): simply define \(f(x) = f_x(Y)\). It remains to show that it satisfies (2.a).

Suppose that for every x \in X, \(f_x \notin F_x. \) We find x \neq y in X with x \in f_y and y \in
\(f_x. \) Let \(\mathcal{Y}_x\) denote the finite set of subsets of X such that \(f_x = \bigcap \{f_y(Y) | Y \in \mathcal{Y}_x\}. \)

We first choose a function \(h: \mathbb{N}^2 \to X\) by induction. For i \in \mathbb{N} let \(H_i = \{h(i', j) | i' < i\}\) and we assume that \((H_i)\) is finite. Every y \in X has an “ith
characteristic function” \(c_y: (\bigcup_{x \in H_i} y_x) \to 2\) defined by \(c_y(Y) = 1\) if and only if
\(y \in Y. \) If c_1, ..., c_n is a listing of the possible ith characteristic functions for terms
of \(X - H_i\), for each j \leq n choose y \in X - H_i with \(c_y = c_j; \) then define \(h(i, j) = y. \)
For j > n, define \(h(i, j) = h(i, 1). \)

Let \(H = \text{range of } h\) and \(\mathcal{Y} = \bigcup \{\mathcal{Y}_x | x \in H\}. \) If x \in H, x \in H_i for a minimal i; define \(\mathcal{Y}^*_x = \mathcal{Y}_x - \bigcup \{\mathcal{Y}_z | z \in H_{i+1}\}. \) For Y \in \mathcal{Y} there is a unique i = i_Y \in \mathbb{N} such that \(Y \in \mathcal{Y}^*_x\) for some x \in H_i.

If Y \neq Z in \(\mathcal{Y}\) choose aY, Z in either Y - Z or Z - Y; then let A = \(\{aY, Z | Y \neq Z \in \mathcal{Y}\}. \) For Y \in \mathcal{Y}, let A_Y = A \cap Y. \) Observe that A is countable, A = \(\bigcup \{A_Y | Y \in \mathcal{Y}\}, \) and that if Y \neq Z in \(\mathcal{Y}, \) then A_Y \neq A_Z. \) For each x \in H, let \(A_x = \{A_Y | Y \in \mathcal{Y}_x\}, \) and define k: \(\mathbb{N}^2 \to (X \times \mathcal{B})\) by \(k(i, j) = (x, A_x)\) if \(h(i, j) = x. \)

Choose y \in X such that y > x for all x \in H and k_y = k. Since \(\mathcal{Y}^*_x\) is finite, so is \(\mathcal{Y}^*_y = \{Y \in \mathcal{Y} | A_Y = (A \cap Z) \text{ for some } Z \in \mathcal{Y}^*_y\}. \) Thus there is an i \in \mathbb{N} such that
\(i > i_Y \text{ for all } Y \in \mathcal{Y}^*_y. \) Since y \in X - H, there is x \in H_{i+1} - H_i such that the ith
characteristic functions of x and y are the same.
We show that \( x \in f_y \) and \( y \in f_x \), thus proving our theorem.

Recall that \( f_z = \bigcap \{ f_x(Y) \mid Y \in \mathcal{Y}_x \} \). If \( Y \in (\mathcal{Y}_x - \mathcal{Y}_y^*) \), then \( i_Y \leq i \) and, since \( x \) and \( y \) have the same \( i \)th characteristic function, \( x \in Y \) if and only \( y \in Y \). Thus, by (a), \( y \in f_z(Y) \). If \( Y \in \mathcal{Y}_y^* \), then since \( k_y = k \), \( (x, \mathcal{A}_z) \in \) \( \langle \text{range } k_y \rangle \). Since \( A_Y = A \), \( A_Y = (Y \cap A) \in \mathcal{A}_z \). By (b), since \( x < y \) and \( y \in g_x(Y) \), \( y \in f_x(Y) \). So \( y \in f_z \).

To show that \( x \in f_y \), suppose that \( \forall y \in f_y \). By our choice of \( i \), if \( A \cap Y = A_Z \) for some \( Z \in \mathcal{Y} \), then \( i_Z < i \); so \( A_y \cap Y = A \cap Y \in \mathcal{A}_x \). Thus \( y \notin g_x(Y) \) since \( (x, \mathcal{A}_z) \) is the only point in the range of \( k \) having \( x \) as its first coordinate. So, by (c), \( x \in f_y(Y) \) and \( x \in f_y \) and our proof is complete.

**Lemma 2.** The existence of a first countable, \( T_4 \), noncollectionwise Hausdorff space implies the answer to Question 2 is yes with the \( F_x \)'s each being countable.

**Proof.** Let \( S \) be the space and \( X \) the closed discrete set of points in \( S \) which cannot be separated; we assume without loss of generality, that the points of \( X - S \) are isolated. For each \( x \in X \), choose an open basis \( U_1(x) \supset U_2(x) \supset \cdots \) for the topology at \( x \) with \( U_1(x) \cap X = \{x\} \). For each \( x \in X \), let \( f_n(x) = \{y \in X - \{x\} \mid U_n(x) \cap U_n(y) \neq \emptyset \} \). Then, if \( F_X = \{f_n(x) \mid n \in \mathbb{N} \} \), \( \{F_x \mid x \in X \} \) has the desired properties.

**Comments on cardinalities.** It has long been known that there is a model for ZFC in which there is a \( T_4 \), noncollectionwise Hausdorff space of countable character. In another model, all \( T_4 \), noncollectionwise Hausdorff spaces have character greater than \( c \).

If \( X \) and \( \{F_x \mid x \in X \} \) yield a yes answer to Question 2, the manifold \( K \) constructed as in the proof of Lemma 1 is a \( T_4 \), noncollectionwise Hausdorff space and each vertex \( x \) has character \( \leq |F_x| \) (or countable if \( F_x \) is countable). By the proofs of Lemmas 1 and 2, the existence of \( X \) and \( \{F_x \mid x \in X \} \) yielding a yes answer to Question 2 with each \( F_x \) countable is equivalent to the existence of a \( K \) yielding a yes answer to Question 3 of countable character is equivalent to the existence of any \( T_4 \), noncollectionwise Hausdorff space of countable character; and all are undecidable in ZFC. Similarly the existence of \( X \) and \( \{F_x \mid x \in X \} \) yielding a yes answer to Question 2 with each \( F_x \) of cardinality \( \leq c \) or a \( K \) yielding a yes answer to Question 3 of character \( \leq c \) is undecidable in ZFC.

The \( X \) of our Theorem has cardinality \( c \). The cardinality of a particular \( F_x \) depends on the choice of \( \{k_y \mid y \in X \} \), but it could be as much as \( 2^c \). There is a consistency example with the cardinality of \( X = \omega_1 \) and all \( F_x \) are countable. But for a real example, one cannot hope to do much better. However, none of these solutions of Question 2 are apt to lead to a solution of Question 1 since Dowker proves that any \( X \) and \( F \) yielding a yes answer to Question 1 must have the cardinality of \( X \) greater than \( \omega_2 \). For completeness I give some of Dowker's results.

**Comments on Question 1 (Dowker).** Suppose that \( X \) and \( F \) yield a yes answer to Question 1 and that the cardinality \( \alpha \) of \( X \) is minimal for such to exist. Then \( Y \in F \) implies \( Y \) has cardinality \( \alpha \) since \( \{A \cap Y \mid A \in F \} \) would be an "\( F \)" set for \( Y \). Let \( \beta \) be the minimal cardinality of a subset of \( X \) whose complement is not in \( F \). By (1.a), no nested sequence of members of \( F \) has empty intersection. If \( Y = \{y_\gamma \mid \gamma < \beta \} \subset X \) and for each \( \delta < \beta \), \( Y_\delta = X - \{y_\gamma \mid \gamma < \delta \} \), then \( \{Y_\delta \mid \delta < \beta \} \) is a nested sequence of members of \( F \); hence \( \beta < \alpha \). Since there is such a \( Y \) with \( (X - Y) \notin F \), and \( \beta < \alpha \) implies \( Y \notin F \), \( F \) is not an ultrafilter. Assuming \( (X - Y) \notin F \)
Let $f: X \to F$ satisfy (1.6) and, for $\delta < \beta$, define $Y^*_\delta = Y_\delta \cap f(Y_\delta)$. Then $Y^*_\delta \in F$ and $\bigcap_{\delta < \beta} Y^*_\delta = \emptyset$. It is impossible to have $\beta \leq \omega$ since then $Z_\delta = \bigcap_{\gamma \leq \delta} Y^*_\gamma \in F$ and $\{Z_\delta \mid \delta < \beta\}$ is a nested sequence with empty intersection. Hence $\omega < \beta < \alpha$ and $\alpha \geq \omega_2$.

Bibliography


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