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EVERY MAPPING OF THE PSEUDO-ARC ONTO ITSELF IS A NEAR HOMEOMORPHISM

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ABSTRACT. The theorem stated in the title is proven. It follows then that the homeomorphism group of the pseudo-arc is dense in the space of continuous functions from the pseudo-arc onto itself.

In general, little is known about the structure of the space of homeomorphisms of a space onto itself. Beverly Brechner [**Br1**, **Br2**] and Wayne Lewis [**Lw**] have published some results concerning the homeomorphism group of the pseudo-arc. We show that the homeomorphism group of the pseudo-arc is dense in the space of continuous functions from the pseudo-arc onto the pseudo-arc.

A continuum is a compact connected metric space. The pseudo-arc is a hereditarily indecomposable chainable continuum. For the definition and properties of crookedness the reader should consult Bing [**B**]. Theorems A and B are due to Bing [**B**].

DEFINITION. Suppose that X and Y are continua and $f: X \to Y$ is continuous. Then f is said to be a *near homeomorphism* if it is true that for each positive number ε there exists a homeomorphism $h: X \to Y$ so that $d(f(x), h(x)) < \varepsilon$ for all $x \in X$.

The theorem stated in the title follows from Theorem 1.

DEFINITION. Suppose that N is a function from the set of positive integers $\{1, \ldots, r\}$ into the set of positive integers $\{1, \ldots, s\}$. Then N *is a pattern* means that for each $1 \leq i < r$, $|N(i+1) - N(i)| \leq 1$.

DEFINITION. If N is a pattern then *the chain D follows the pattern N in the chain C* means that if $D = d_1, \ldots, d_r$, and $C = c_1, \ldots, c_s$, then $N: \{1, \ldots, r\} \to \{1, \ldots, s\}$ and $d_i \subset c_{N(i)}$.

DEFINITION. The chain E is a *consolidation* of the chain D if each link of E is the union of a subcollection of D and each link of D is a subset of a link of E.

DEFINITION. The chain C is a *chain from the point P to the point Q* means that P belongs to an end link of C and to no other link of C and Q belongs to the other end link of C and to no other link.

Notation. If H and K are sets then let

$$d(H, K) = \operatorname{glb}\{d(x, y) | x \in H, y \in K\}.$$

THEOREM A (BING [**B**]). *Suppose that $N: \{1, \ldots, n\} \to \{1, \ldots, r\}$ is a pattern, N is onto, $N(1) = 1$, $N(n) = r$, D_1, $D_2, \ldots$ is a sequence of chains from the point P to the point Q such that D_1 has r links, D_{i+1} is crooked in D_i for each positive*

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integer i, and $\operatorname{mesh}(D_i) \leq 1/i$. *Then there is an integer j and a chain E from P to Q such that E is a consolidation of the chain D_j and E follows pattern N in D_1.*

THEOREM B (BING [**B**]). *Suppose that M_n $(n = 1, 2)$ is a compact closed set; $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of positive numbers with a finite sum; and $X_{n,1}$, $X_{n,2}, \ldots$ is a sequence of chains such that for each positive integer i, (1) $X_{n,i}$ covers M_n, (2) each element of $X_{n,i}$ intersects M_n, (3) no element of $X_{n,i}$ has a diameter of more than ε_i, and (4) if the jth element of $X_{n,i+1}$ intersects the kth element of $X_{n,i}$, then the distance between the jth element of $X_{m,i+1}$ $(m = 1, 2)$ and the kth element of $X_{m,i}$ is less than ε_i. Then there is a homeomorphism T carrying M_1 into M_2.*

LEMMA. *If X and $\tilde{X}$ are indecomposable continua and f is a map from X onto $\tilde{X}$ then there exist two points a and b so that a and b are in different composants of X and $f(a)$ and $f(b)$ are in different composants of $\tilde{X}$.*

PROOF. Let P and Q be two points of $\tilde{X}$ that lie in different composants of $\tilde{X}$. Then if $f^{-1}(P)$ and $f^{-1}(Q)$ are both subsets of the same composant C of X, let R be a point of X lying in a composant of X different from C. Then R and $f^{-1}(Q)$ are in different composants of X and $f(R)$ lies in a composant which either does not contain Q or does not contain P. Assume the former, then let $a = R$ and $b \in f^{-1}(Q)$. In the latter case let $a = R$ and $b \in f^{-1}(P)$.

THEOREM 1. *Suppose that X and $\tilde{X}$ are pseudo-arcs, $f: X \to \tilde{X}$ is a map of X onto $\tilde{X}$ and $\varepsilon > 0$. Then there exists a homeomorphism $h: X \to \tilde{X}$ such that if $x \in X$ then $d(h(x), f(x)) < \varepsilon$.*

PROOF. Let P and Q be two points of X so that P and Q lie in different composants of X and $f(P)$ and $f(Q)$ lie in different composants of $\tilde{X}$. Let $0 < \delta_1 < \varepsilon/4$.

Let C^1, $C^2, \ldots$ be a sequence of chains from $f(P)$ to $f(Q)$ so that for each positive integer i, C^{i+1} is crooked in C^i, $\operatorname{mesh}(C^i) < 1/i$, the closure of each link of C^{i+1} is a subset of a link of C^i, the closures of two links of C^i intersect if and only if the links intersect, and so that C^1 is a δ_1 chain.

Let D^1, $D^2, \ldots$ be a sequence of chains from P to Q so that for each positive integer i, D^{i+1} is crooked in D^i, $\operatorname{mesh}(D^i) < 1/i$, the closure of each link of D^{i+1} is a subset of a link of D^i, the closures of two links of D^i intersect if and only if the links intersect, and so that if $d \in D^1$ then $f(d)$ is a subset of some link of C^1.

Let

$$D^i = d_1^i, d_2^i, \ldots, d_{k_i}^i \quad \text{with } P \in d_1^i, Q \in d_{k_i}^i,$$

and

$$C^i = c_1^i, c_2^i, \ldots, c_{l_i}^i \quad \text{with } f(P) \in c_1^i, f(Q) \in c_{l_i}^i.$$

If $i \in \{1, \ldots, k_1\}$, let $N^1(i)$ denote the first integer n so that $f(d_i^1) \subset c_n^1$. Since C^1 and D^1 are chains from $f(P)$ to $f(Q)$ and P to Q respectively, then N^1 is a pattern and $N^1(1) = 1$, $N^1(k_1) = l_1$.

Let δ_2 be a positive number less than the distance between every two nonintersecting links of C^1 and less than δ_1. Let $B^1 = C^1$.

There is an integer j_2 and a consolidation B^2 of the links of C^{j_2} so that B^2 is a chain from $f(P)$ to $f(Q)$ that follows N^1 in C^1. From the definition of consolidation it follows that the closure of two links of B^2 intersect only if the links themselves do so. Let $0 < \delta_3 < \delta_2$, so that the distance between nonintersecting links of B^2 is at least δ_3 and let C^{j_3} be a $\delta_3/3^3$ chain which refines B^2 so that at least ten links of C^{j_3} are needed to connect two nonintersecting links of B^2, and let N^2 be a pattern of C^{j_3} in B^2. Let $B^3 = C^{j_3}$.

Let $A^1 = D^1$. There is an integer $i_3 > 1$ and a chain A^2 from P to Q that is a consolidation of D^{i_3} that follows N^2 in D^1. Let $0 < \delta_4 < \delta_3$, let D^{i_4} be a $\delta_4/3^4$ chain which refines A^2, and let N^3 be a pattern of D^{i_4} in A^2. Let $A^3 \doteq D^{i_4}$. There is an integer j_4 and a chain B^4 from $f(P)$ to $f(Q)$ that is a consolidation of C^{j_4} so that B^4 follows N^3 in B^3. Let δ_5 be a positive number, $\delta_5 < \delta_4$, so that the distance between nonintersecting links of B^4 is at least δ_5.

Thus by induction we can construct a sequence $A^1, A^2, \ldots$ of chains from P to Q covering X, a sequence $B^1, B^2, \ldots$ of chains from $f(P)$ to $f(Q)$ covering $\tilde{X}$, a decreasing sequence $\delta_1, \delta_2, \ldots$ of positive numbers so that if k is an even positive integer:

(i) the distance between nonintersecting links of B^k is at least δ_{k+1},
(ii) $B^{k+1} = C^{j_{k+1}}$ is a $\delta_{k+1}/3^{k+1}$ chain which refines B^k so that at least ten links of B^{k+1} are needed to connect two nonintersecting links of B^k,
(iii) B^{k+1} follows pattern N^k in B^k,
(iv) A^{k+1} follows pattern N^{k+1} in A^k,
(v) A^{k+2} is a consolidation of $D^{j_{k+3}}$ which is a $\delta_{k+2}/3^{k+2}$ chain which refines A^{k+2} and follows pattern N^{k+2} in A^{k+1}, and
(vi) B^{k+2} follows pattern N^{k+1} in B^{k+1}.

Note that B^{k+1} and A^k have the same number of links. Also, for k even, A^{k-1} and A^k are $\delta_k/3^k$ chains and B^{k+1} and B^{k+2} are $\delta_{k+1}/3^{k+1}$ chains, and if k is an even integer then two nonintersecting links of B^k are at least δ_{k+1} apart.

We have now defined a sequence of chains which satisfy the hypothesis of Theorem B. So there exists a homeomorphism h mapping X into $\tilde{X}$. We shall outline the construction of the homeomorphism and state some of its properties which the reader may verify by consulting Bing's proof of Theorem B.

Suppose $x \in X$. Then let r_i^x denote an integer n so that $x \in a_n^i$. Thus $x = \bigcap_{i=1}^{\infty} a^i_{r_i^x}$. For each positive integer i let $P^{i+1}_{r_i^x}$ be a point in $b^{i+1}_{r_i^x}$. It follows from the construction that $\{P^{i+1}_{r_i^x}\}_{i=1}^{\infty}$ is a Cauchy sequence. Also since $\lim_{i\to\infty} \operatorname{mesh} B^i = 0$ it follows that if $Q^{i+1}_{r_i^x} \in b^{i+1}_{r_i^x}$, then $\{Q^{i+1}_{r_i^x}\}_{i=1}^{\infty}$ has the same sequential limit as $\{P^{i+1}_{r_i^x}\}_{i=1}^{\infty}$. Thus if $h(x) = \lim_{i\to\infty} P^{i+1}_{r_i^x}$, then h is well defined and is the required transformation. Further we have that

$$(*) \qquad \lim_{i\to\infty} d(h(x), b^{i+1}_{r_i^x}) = 0$$

and in particular $d(h(x), b^2_{r_1^x}) < \delta_1 < \varepsilon/4$.

The fact that h is a homeomorphism of X into $\tilde{X}$ follows from condition $(*)$ together with the ten-link condition (condition (ii)). For each positive integer k, A^k is a chain from P to Q and B^k is a chain from $f(P)$ to $f(Q)$. Thus $h(P) = f(P)$ and $h(Q) = f(Q)$ and hence h is onto.

We now need to prove the following claim.

CLAIM. For each $x \in X$, $d(f(x), h(x)) < \varepsilon$.

PROOF. Suppose x is a point of X and $x \in d_i^1$. Then $f(x) \in f(d_i^1)$ and $f(d_i^1) \subset c^1_{N(i)}$. Let $x \in a^1_{r_1^x}$, and since $A^1 = D^1$, r_1^x is one of $i-1$, i, or $i+1$. So $N^1(r_1^x)$ is one of $N(i-1)$, $N(i)$, or $N(i+1)$. Now B^2 follows N^1 in C^1 so

$$b^2_{r_1^x} \subset c^1_{N^1(r_1^x)} \subset c^1_{N(i-1)} \cup c^1_{N(i)} \cup c^1_{N(i+1)}.$$

Thus, from above,

$$d(b^2_{r_1^x}, h(x)) < \frac{\varepsilon}{4}, \quad d(h(x), c^1_{N(i)} < 2\frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

So $d(h(x), f(x)) < \varepsilon/2 + \varepsilon/4 < \varepsilon$. This proves the theorem.

Comment. Recently Wayne Lewis has announced similar results.

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