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LARGE HIGHLY POWERFUL NUMBERS ARE CUBEFUL

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ABSTRACT. A number $n = \prod_{i=1}^{k} p_i^E$ is called *highly powerful* if the product of the exponents $E(p_i)$ of the primes is larger than that of any smaller number. If $p_k > 19$, $E(p_k) = 3$. Further, we have developed an algorithm which finds all highly powerful numbers with $E(p_k) \neq 3$, and we list the 19 highly powerful numbers with $E(p_k) = 2$.

Let the *prodex* of n be the product of the exponents of the primes when n is written in standard form. M. V. Subbarao has called a number *highly powerful* if its prodex is larger than that of any smaller number. Assume that $n = \prod_{i=1}^{k} p_i^E$ is highly powerful. Then it is clear that p_i is the ith prime, the exponents $E = E(p_i)$ are nonincreasing, $E(p_k) \geq 2$ and $E(p_{k-1}) \geq 3$ (since $p_{k-1}^4 < p_{k-1}^2 p_k^2$). The theorem of the title asserts that if $p_k > N$, then $E(p_k) \geq 3$. Further, we have developed an algorithm which finds all highly powerful numbers having $E(p_k) \neq 3$. The nineteen highly powerful numbers with $E(p_k) = 2$ are listed in Table 1.

Throughout the paper, $n = \prod_{i=1}^{k} p_i^E$ will be assumed to be a highly powerful number and the exponent of p_i will be written $E(p_i)$. In the proof of Theorem 1, we will introduce a one-line notation which will be used throughout the paper.

THEOREM 1. *If n is highly powerful and $p_k > 13$, then $E(p_k) \leq 3$.*

PROOF. Assuming the contrary ($E(p_k) \geq 4$), we will show that n is not highly powerful by exhibiting a smaller number whose prodex is equal to or greater than prodex(n).

First we show that $E(2) \geq 10$. If $E(2) \leq 9$, multiply by 8 and divide by p_k. The changes in the prodex will be $(E(2) + 3)/E(2)$ and $E(p_k)/(E(p_k) - 1)$.

In our one-line notation, we will first write the conclusion, followed by the ratio of the new number to n, and an inequality or equality, the left side of which will be a lower bound for the changes that increase the prodex, the right side of which will be an upper bound for the changes that decrease the prodex.

Thus, in the case above, we have

$$E(2) \geq 10 \qquad 2^3/p_k \qquad 12/9 = 4/3$$

and the new number has the same or a larger prodex. In the remaining case, multiply by p_{k+1}^2 and divide by $2p_{k-1}p_k$.

end proof $\qquad p_{k+1}^2/2p_{k-1}p_k \qquad 2 > (10/9)(4/3)^2$

It remains to prove that $p_{k+1}^2 < 2p_{k-1}p_k$ when $p_k \geq 17$. This can be verified directly for $p_k = 17$, 19, 23 and 29. If $p_{k-1} > (31/37)29$, $p_k \leq (37/31)p_{k-1}$. This

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and similar statements follow from Table 3. Thus, $p_{k+1}^2 < (37/31)^3 p_{k-1} p_k$ and, since $(37/31)^3 < 2$, our proof of Theorem 1 is complete, with the words "end proof" signaling that a contradiction has been reached.

The main result.

THEOREM 2. *If* $p_k > 2190$, *then* $E(p_k) = 3$.

PROOF. Let q_1 be the largest prime less than $p_k^{1/4}$, q_2 the smallest prime greater than $p_k^{1/3}$, and q_3 and q_4 the smallest consecutive primes such that $q_1 q_3$ is greater than p_k.

Since $p_k^{1/4} > 5$, $q_1 > (7/11) p_k^{1/4}$. This follows since there is always a prime between $(7/11)x$ and x if $x > 5$. Similarly, since $p_k^{1/3} > (13/17)11$, $q_2 < (17/13) p_k^{1/3}$. Since $p_k^{3/4} > (523/541)331$, $q_3 < (541/523) p_k / q_1 < (11/7)(541/523) p_k^{3/4}$ and $q_4 < (11/7)(541/523)^2 p_k^{3/4}$. From this it follows that

$$q_2 q_3 q_4 < (17/13)(11/7)^2 (541/523)^3 p_k^{11/6} < p_k^2.$$

Assume $E(p_k) = 2$. Then since $q_1 < p_k^{1/4}$,

$$E(q_1) \geq 9 \qquad q_1^8 / p_k^2 \qquad 16/8 = 2.$$

Since $q_1 q_3 > p_k$,

$$E(q_3) = 3 \qquad p_k / q_1 q_3 \qquad 3/2 = (9/8)(4/3).$$

Since $q_2 q_3 q_4 < p_k^2$,

$$E(q_2) \geq 9 \qquad q_2 q_3 q_4 / p_k^2 \qquad (9/8)(4/3)^2 = 2.$$

Since $q_2^3 > p_k$,
end proof $\qquad p_k / q_2^3 \qquad 3/2 = 9/6.$

THEOREM 3. *If* $631 \leq p_k \leq 2213$, *or* $83 \leq p_k \leq 113$, *then* $E(p_k) = 3$.

The argument used above applies for the following:

range of p_k	q_1	q_2	q_3	q_4
1847–2213	5	17	443	449
1361–1861	5	13	373	379
907–1327	5	11	269	271
631–887	5	11	179	181
97–113	3	5	41	43
83, 89	3	5	31	37

THEOREM 4. *If* $p_k = 37$ *or* $p_k \geq 53$, $p_k \neq 67$, *then* $E(p_k) = 3$.

PROOF. For the values of p_k not yet considered, we extend the previous argument.

Let $q_1^4 < p_k$, $q_1 q_6 > p_k$, $q_2 q_6 q_7 < p_k^2$, $q_2 q_4 > p_k$, $q_3 q_4 q_5 < p_k^2$, and $q_3^3 > p_k$. Assume $E(p_k)$. Then since $q_1^8 < p_k^2$,

$$E(q_1) \geq 9 \qquad q_1^8 / p_k^2 \qquad 16/8 = 2.$$

Since $q_1q_6 > p_k$,

$$E(q_6) = 3 \qquad p_k/q_1q_6 \qquad 3/2 = (9/8)(4/3).$$

Since $q_2q_6q_7 < p_k^2$,

$$E(q_2) \geq 9 \qquad q_2q_6q_7/p_k^2 \qquad (9/8)(4/3)^2 = 2.$$

Since $q_2q_4 > p_k$,

$$E(q_4) = 3 \qquad p_k/q_2q_4 \qquad 3/2 = (9/8)(4/3).$$

Since $q_3q_4q_5 < p_k^2$,

$$E(q_3) \geq 9 \qquad q_3q_4q_5/p_k^2 \qquad (9/8)(4/3)^2 = 2.$$

Since $q_3^3 > p_k$,

end proof $\qquad p_k/q_3^3 \qquad 3/2 = 9/6,$

This argument applies for the following:

range of p_k	q_1	q_2	q_3	q_4	q_5	q_6	q_7
487–631	3	5	11	127	131	211	223
373–487	3	5	11	101	103	163	167
293–379	3	5	11	79	83	127	131
223–283	3	5	7	59	61	97	101
163–211	3	5	7	43	47	71	73
127–157	3	5	7	37	41	53	59
73, 79	2	3	5	29	31	41	43
71, 73	2	3	5	29	31	37	41
59, 61	2	3	5	23	29	31	37
53	2	3	5	19	23	29	31
37	2	3	5	13	17	19	23

THEOREM 5. *If* $p_k = 67, 47, 43, 41, 31, 29,$ *or* 23, *then* $E(p_k) = 3$.

Assume $E(67) = 2$. Then

	$E(2) \geq 13$	$2^{12}/67^2$	$24/12 = 2$
	$E(37) = 3$	$67/2 \cdot 37$	$3/2 > (13/12)(4/3)$
	$E(3) \geq 9$	$3^4 37/67^2$	$(12/8)(4/3) = 2$
	$E(23) = 3$	$67/3 \cdot 23$	$3/2 = (9/8)(4/3)$
	$E(5) \geq 9$	$5 \cdot 23 \cdot 29/67^2$	$(9/8)(4/3)^2 = 2$
end proof		$67/5^3$	$3/2 = 9/6$

Assume $E(p_k) = 2$ when $p_k = 47$ and $q_6 = 29$, or when $41 \le p_k \le 43$ and $q_6 = 23$. Then

	$E(2) \ge 11$	$2^{10}/p_k^2$	$20/10 = 2$
	$E(q_6) = 3$	$p_k/2q_6$	$3/2 > (11/10)(4/3)$
	$E(7) \le 5$	$p_k/7^2$	$3/2 = 6/4$
	$E(3) \ge 9$	$3^2 7 q_6/p_k^2$	$(10/8)(6/5)(4/3) = 2$
	$E(17) = 3$	$p_k/3 \cdot 17$	$3/2 = (9/8)(4/3)$
	$E(5) \ge 9$	$5 \cdot 17 \cdot 19/p_k^2$	$(9/8)(4/3)^2 = 2$
end proof		$p_k/5^3$	$3/2 = 9/6$

Assume $E(31) = 2$. Then

	$E(2) \ge 10$	$2^9/31^2$	$18/9 = 2$
	$E(3) \ge 7$	$3^6/31^2$	$12/6 = 2$
	$E(2) \le 14$	$31/2^5$	$3/2 = 15/10$
	$E(17) = 3$	$31/2 \cdot 17$	$3/2 > (10/9)(4/3)$
	$E(7) \ge 5$	$2^3 7 \cdot 17/31^2$	$(17/14)(5/4)(4/3) > 2$
	$E(5) = 5$	$31/5 \cdot 7$	$3/2 = (6/5)(5/4)$
	$E(11) \ge 53$	$5 \cdot 11 \cdot 17/31^2$	$(6/5)(5/4)(4/3) = 2$
end proof		$31/3 \cdot 11$	$3/2 > (7/6)(5/4)$

Assume $E(29) = 2$. Then

	$E(2) \ge 10$	$2^9/29^2$	$18/9 = 2$
	$E(17) = 3$	$29/2 \cdot 17$	$3/2 > (10/9)(4/3)$
	$E(7) \ge 5$	$7^2 17/29^2$	$(6/4)(4/3) = 2$
	$E(5) = 53$	$29/5 \cdot 7$	$3/2 = (6/5)(5/4)$
	$E(3) \ge 9$	$3^2 5 \cdot 17/29^2$	$(10/8)(6/5)(4/3) = 2$
	$E(11) = 5$	$29/3 \cdot 11$	$3/2 = (9/8)(4/3)$
end proof		$5 \cdot 11 \cdot 13/29^2$	$(6/5)(4/3)^2 > 2$

Assume $E(23) = 2$. Then

	$E(2) \ge 10$	$2^9/23^2$	$18/9 = 2$
	$E(13) = 3$	$23/2 \cdot 13$	$3/2 > (10/9)(4/3)$
	$E(5) \le 5$	$23/5^2$	$3/2 = 6/4$
	$E(7) = 5$	$5 \cdot 7 \cdot 13/23^2$	$(6/5)(5/4)(4/3) = 2$
	$E(2) \le 11$	$23/2^2 7$	$3/2 = (12/10)(5/4)$
end proof		$2^3 5 \cdot 13/23^2$	$(14/11)(6/5)(4/3) > 2$

This completes the proof that $E(p_k) = 3$ if $p_k > 19$.

Highly powerful numbers for which $E(p_k) = 2$. The following arguments show that there are no highly powerful numbers with $E(p_k) = 2$ beyond those listed in Table 1.

Assume $E(19) = 2$. Then

$E(2) \geq 9$	$2^8/19^2$	$16/8 = 2$
$E(11) = 3$	$19/2 \cdot 11$	$3/2 = (9/8)(4/3)$
$E(3) \leq 8$	$19/3^3$	$3/2 = 9/6$
$E(3) \geq 7$	$3^3 11/19^2$	$(9/6)(4/3) = 2$
$E(5) \leq 5$	$19/5^2$	$3/2 = 6/4$
$E(5) = 5$	$5^2 11/19^2$	$(6/4)(4/3) = 2$
$E(7) \leq 4$	$19/3 \cdot 7$	$3/2 > (7/6)(5/4)$
$E(7) = 4$	$7^3/19^2$	$6/3 = 2$
$E(2) \geq 11$	$2^5 11/19^2$	$(15/10)(4/3) = 2$
$E(2) = 11$	$19/2^2 5$	$3/2 = (12/10)(5/4)$

Two possible solutions: $2^{11}3^{7,8}5^5 7^4 11^3 13^3 17^3 19^2$.

Assume $E(17) = 2$. Then

$E(2) \geq 9$	$2^8/17^2$	$16/8 = 2$
$E(11) = 3$	$17/2 \cdot 11$	$3/2 = (9/8)(4/3)$
$E(3) \geq 6$	$3^5/17^2$	$10/5 = 2$
$E(3) \leq 7$	$17/2 \cdot 3^2$	$3/2 = (9/8)(8/6)$
$E(5) \leq 5$	$17/5^2$	$3/2 = 6/4$
$E(5) = 5$	$5^2 11/17^2$	$(6/4)(4/3) = 2$
$E(7) \leq 4$	$17/3 \cdot 7$	$3/2 = (6/5)(5/4)$
$E(7) = 4$	$3 \cdot 7 \cdot 11/17^2$	$(8/7)(4/3)^2 > 2$
$E(2) \leq 11$	$17/2^2 5$	$3/2 = (12/10)(5/4)$
$E(2) \geq 10$	$2^5 3^2/17^2$	$(14/9)(9/7) = 2$
not $2^{10}3^6$	$2^5 3^2/17^2$	$(15/10)(8/6) = 2$

Three possible solutions:

$$2^{11}3^6 5^5 7^4 11^3 13^3 17^2, \qquad 2^{10,11}3^7 5^5 7^4 11^3 13^3 17^2.$$

Assume $E(13) = 2$. Then

$E(2) \leq 11$	$13/2^4$	$3/2 = 12/8$
$E(3) \leq 8$	$13/3^3$	$3/2 = 9/6$
$E(5) \geq 4$	$5^3/13^2$	$6/3 = 2$
$E(5) \leq 5$	$13/5^2$	$3/2 = 6/4$

Suppose $E(5) = 5$,

$E(3) = 5$	$13/3 \cdot 5$	$3/2 = (6/5)(5/4)$
$E(2) \geq 10$	$2 \cdot 3^4/13^2$	$(10/9)(9/5) = 2$
$E(7) = 3$	$13/2 \cdot 7$	$3/2 > (10/9)(4/3)$
end subproof	$2^3 3 \cdot 7/13^2$	$(14/11)(6/5)(4/3) > 2$

Thus, $E(5) = 4$ and

$E(2) \geq 9$	$2^5 5/13^2$	$(13/8)(5/4) > 2$
$E(7) = 3$	$13/2 \cdot 7$	$3/2 = (9/8)(4/3)$
$E(2) = 11$	$2^2 5 \cdot 7/13^2$	$(12/10)(5/4)(4/3) = 2$
$E(3) \geq 7$	$3^2 5 \cdot 7/2 \cdot 13^2$	$(8/6)(5/4)(4/3) > (11/10)2$
$E(3) = 7$	$13/2 \cdot 3^2$	$3/2 > (11/10)(8/6)$

One possible solution: $2^{11}3^7 5^4 7^3 11^3 13^2$.

Assume $E(11) = 2$. Then

$E(2) \geq 7$	$2^6/11^2$	$12/6 = 2$
$E(3) \leq 8$	$11/3^3$	$3/2 = 9/6$
$E(5) \leq 5$	$11/5^2$	$3/2 = 6/4$
$E(7) \leq 4$	$11/2 \cdot 7$	$3/2 > (7/6)(5/4)$

Suppose $E(7) = 4$,

	$E(2) \leq 8$	$11/2 \cdot 7$	$3/2 = (9/8)(4/3)$
	$E(3) \geq 6$	$2^2 3^3/11^2$	$(10/8)(8/5) = 2$
	$E(5) = 4$	$11/3 \cdot 5$	$3/2 = (6/5)(5/4)$
	$E(3) \geq 7$	$2^3 3 \cdot 5/11^2$	$(11/8)(7/6)(5/4) > 2$
	$E(2) = 8$	$2^3 3 \cdot 5/11^2$	$(10/7)(9/8)(5/4) > 2$
end subproof		$2^2 5/3 \cdot 7$	$(10/8)(5/4) > (7/6)(4/3)$

Thus, $E(7) = 3$ and

$E(2) \geq 9$	$2^4 7/11^2$	$(12/8)(4/3) = 2$
$E(5) \geq 4$	$3 \cdot 5 \cdot 7/11^2$	$(9/8)(4/3)^2 = 2$
$E(3) \leq 6$	$11/2^2 3$	$3/2 = (9/7)(7/6)$

Suppose $E(5) = 5$,

	$E(3) = 5$	$11/3 \cdot 5$	$3/2 = (6/5)(5/4)$
	$E(2) \geq 10$	$2 \cdot 11/5^2$	$(10/9)(3/2) = 5/3$
end subproof		$3^2/2 \cdot 5$	$7/5 > (10/9)(5/4)$

Thus, $E(5) = 4$ and

$E(3) = 6$	$3 \cdot 5 \cdot 7/11^2$	$(6/5)(5/4)(4/3) = 2$
$E(2) = 9$	$11/2^2 3$	$3/2 = (10/8)(6/5)$

One possible solution: $2^9 3^6 5^4 7^3 11^2$.

Assume $E(7) = 2$. Then

$E(3) \geq 4$	$3^3/7^2$	$6/3 = 2$
$E(3) \leq 5$	$7/3^2$	$3/2 = 6/4$
$E(2) \geq 7$	$2^4 3/7^2$	$(10/6)(6/5) = 2$
$E(2) \leq 8$	$7/2^3$	$3/2 = 9/6$
$E(5) \leq 4$	$7/2 \cdot 5$	$3/2 > (7/6)(5/4)$
not $3^4 5^3$	$3^2 5/7^2$	$(6/4)(4/3) = 2$
not $3^5 5^4$	$2 \cdot 7/3 \cdot 5$	$(9/8)(3/2) > (5/4)(4/3)$

Four possible solutions:

$$2^{7,8} 3^5 5^3 7^2, \qquad 2^{7,8} 3^4 5^4 7^2.$$

Assume $E(5) = 2$. Then

$E(2) \leq 8$	$5/2^3$	$3/2 = 9/6$
$E(2) \geq 5$	$2^4/5^2$	$8/4 = 2$
$E(3) \leq 5$	$5/3^2$	$3/2 = 6/4$
not $2^{6-8} 3^5$	$5/2 \cdot 3$	$3/2 = (6/5)(5/4)$
not $2^5 3^4$	$2^3 3/5^2$	$(8/5)(5/4) = 2$
not $2^{5,6} 3^3$	$2^3 3/5^2$	$(9/6)(4/3) = 2$
not $2^8 3^3$	$3/2^2$	$4/3 = 8/6$

Five possible solutions:

$$2^7 3^3 5^2, \qquad 2^5 3^5 5^2, \qquad 2^{6-8} 3^4 5^2.$$

Assume $E(3) = 2$. Then

$E(2) \geq 4$	$2^3/3^2$	$6/3 = 2$
$E(2) \leq 5$	$3/2^2$	$3/2 = 6/4$

Two possible solutions: $2^{4,5} 3^2$.

All highly powerful numbers with $E(p_k) = 2$ are listed below:

TABLE 1

The 19 highly powerful numbers which are not cubeful

2^2	$2^8 3^4 5^2$	$2^{11} 3^6 5^5 7^4 11^3 13^3 17^2$
$2^4 3^2$	$2^7 3^5 5^3 7^2$	$2^{10} 3^7 5^5 7^4 11^3 13^3 17^2$
$2^5 3^2$	$2^7 3^4 5^4 7^2$	$2^{11} 3^7 5^5 7^4 11^3 13^3 17^2$
$2^7 3^3 5^2$	$2^8 3^5 5^3 7^2$	$2^{11} 3^7 5^5 7^4 11^3 13^3 17^3 19^2$
$2^6 3^4 5^2$	$2^8 3^4 5^4 7^2$	$2^{11} 3^8 5^5 7^4 11^3 13^3 17^3 19^2$
$2^5 3^5 5^2$	$2^9 3^6 5^4 7^3 11^2$	
$2^7 3^4 5^2$	$2^{11} 3^7 5^4 7^3 11^3 13^2$	

On first examination we had 21 possible solutions. These cases were tested by a small program which generated 2682 numbers including all highly powerful numbers less than or equal to the largest or these cases, namely

$$2^{11}3^8 5^5 7^4 11^3 13^3 17^3 19^2 \quad (5.23 \cdot 10^{26}).$$

Each number generated by the program was tested to see if it were a smaller number with greater or equal prodex when compared in turn with each of the 21 cases. Two of the 21 cases were found not to be highly powerful. We perfected our arguments, which now eliminate these two cases.

Highly powerful numbers for which $E(p_k) > 3$. First we give a slight improvement of Theorem 1.

THEOREM 6. *If* n *is highly powerful and if* $p_k > 7$, *then* $E(p_k) \le 3$.

PROOF. Assume $p_k = 13$ and $E(13) \ge 4$. Then

	$E(2) \ge 10$	$2^3/13$	$12/9 = 4/3$
	$E(5) \ge 5$	$5^2/2 \cdot 13$	$6/4 > (10/9)(4/3)$
end proof		$17^3/5 \cdot 7 \cdot 11 \cdot 13$	$3 > (5/4)(4/3)^3$

Assume $p_k = 11$ and $E(11) \ge 4$. Then

	$E(2) \ge 10$	$2^3/11$	$12/9 = 4/3$
	$E(3) \ge 7$	$3^3/11$	$8/6 = 4/3$
	$E(5) = 4$	$13^3/2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	$3 > (10/9)(7/6)(5/4)(4/3)^2$
	$E(2) \ge 16$	$2 \cdot 5/11$	$(16/15)(5/4) = 4/3$
end proof		$13^2/2^4 11$	$2 > (16/12)(4/3)$

Subbarao and G. E. Hardy [3] have listed seventeen highly powerful numbers with $p_k = 7$, $E(p_k) = 4$ and nine with $p_k = 5$, $E(p_k) = 4$, and are sure that the list is complete.

THEOREM 7. *If* n *is highly powerful and* $p_k > 3$, *then* $E(p_k) \le 4$. *If* $p_k = 3$, $E(p_k) \le 5$.

Assume $p_k = 7$ and $E(7) \ge 5$. Then

	$E(3) = 5$	$11^2/3 \cdot 7^2$	$2 = (6/5)(5/3)$
	$E(2) \le 9$	$3^2/2 \cdot 7$	$7/5 > (10/9)(5/4)$
end proof		$2 \cdot 3/7$	$(10/9)(6/5) > 5/4$

Assume $p_k = 5$ and $E(5) \ge 5$. Then

	$E(2) \ge 9$	$2^2/5$	$10/8 = 5/4$
end proof		$7^2/2 \cdot 5^2$	$2 > (9/8)(5/3)$

Assume $p_k = 3$ and $E(3) \ge 6$. Then

end proof	$5^2/3^3$	$2 = 6/3$

The largest n with $E(p_k) > 4$ is $2^8 3^5$. All n with $p_k \le 3$ are included in the list of the 25 smallest highly powerful numbers in Table 2.

TABLE 2
The 25 smallest highly powerful numbers

2^2	2^7	$2^5 3^3$	$2^6 3^4$	$2^7 3^5$
2^3	$2^4 3^2$	$2^4 3^4$	$2^5 3^5$	$2^9 3^4$
2^4	$2^3 3^3$	$2^6 3^3$	$2^7 3^4$	$2^8 3^5$
2^5	$2^5 3^2$	$2^5 3^4$	$2^6 3^5$	$2^7 3^3 5^2$
2^6	$2^4 3^3$	$2^7 3^3$	$2^8 3^4$	$2^5 3^3 5^3$

Chebyshev-type theorems concerning primes. According to Diamond [**2**], in 1852 Chebyshev proved a result which implies there is always a prime between x and $6x/5$, provided that x is "sufficiently large". If this can be proved for all $x > 10^6$, say, then it is easy to use a small computer to show that it holds for all $x > (5/6)29$. The estimates obtained from the first part of such a computer run are listed in Table 3.

The most recent result for large x that we could find is due to Schoenfeld [**4**, p. 354], which asserts that the open interval $(x, x + x/16597)$ contains a prime for each $x > 2010759.9$. This result and the list of largest gaps between primes up to 10^{12} [**1**] make us sure of the correctness of the entries in Table 3.

TABLE 3
There is always a prime between cx and x
($cx \le p < x$), provided that $x > N(c)$

c	$N(c)$	c	$N(c)$
3/5	2	293/307	223
7/11	5	317/331	307
13/17	11	523/541	331
23/29	17	1327/1361	541
31/37	29	1669/1693	1361
47/53	37	1951/1973	1693
113/127	53	2179/2203	1973
139/149	127	2477/2503	2203
199/211	149	2971/2999	2503
211/223	211	3271/3299	2999

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