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SOME INTERSECTION PROPERTIES OF THE FIBRES OF SPRINGER'S RESOLUTION

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ABSTRACT. Combinatorial results are used to calculate the dimension of the intersection of any two irreducible components of the set in the flag variety fixed by the action of a unipotent element of GL_n whose Jordan decomposition has two blocks. This is then related to the "left cells" of Kazhdan and Lusztig, which are used to construct representations of S_n, the Weyl group of GL_n.

0. In this note, we study the fibres of Springer's resolution [**St**] of the singularities of the unipotent variety in $G = \mathrm{GL}_n(k)$, where k is an algebraically closed field. These fibres are fixed point sets for the action of G on the variety $\mathscr{B}$ parametrizing the complete flags in a vector space of dimension n. We use $\mathscr{B}_u$ to fixed by a unipotent element $u \in \mathrm{SL}_n$.

In general, $\mathscr{B}_u$ has several irreducible components. Suppose that the Jordan decomposition of u has block sizes $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$, where $\lambda_1 + \cdots + \lambda_s = n$. (We refer to this as the *shape* λ, and, by abuse of language, we say that u has *shape* λ.) Then each component of $\mathscr{B}_u$ has dimension $\sum_{i=1}^{s}(i-1)\lambda_i$, and there is one component for each standard Young tableau of shape λ (see [**S**] and below). However, one does not know, in general, the codimension of the intersection of two components; only the "one-hook" case has been done [**V**]. Here, we calculate the codimension when the Jordan form of u has two blocks. This calculation depends on combinatorial techniques exposed in [**LS**]. The precise result is (2.1).

This calculation has two-fold significance. First, it enables one, in this case, to verify a conjecture of Kazhdan and Lusztig [**KL**, 6.3] concerning the configuration of components of $\mathscr{B}_u$; see (4.3). Second, this casts new light on the combinatorial results in [**LS**] which are used to calculate Kazhdan-Lusztig polynomials, because there is no mention of the geometry of the Grassmannian here.

1.0. In this section, we begin to associate (following [**LS**]) components of $\mathscr{B}_u$ (for certain u) to words in the letters α and β (and vice-versa). Let M_a^b denote the set of words made up of a α's and b β's, e.g., $M_2^1 = \{\beta\alpha\alpha, \alpha\beta\alpha, \alpha\alpha\beta\}$. We associate a permutation of $\{1, 2, \ldots, a+b\}$ to each $w \in M_a^b$ by writing $1, \ldots, a$ in order from right to left under the α's, and then $a+1, \ldots, a+b$ in order from right to left under the β's. Thus, $\alpha\alpha\beta\alpha\beta\beta\alpha$ is associated to 4, 3, 7, 2, 6, 5, 1, which can be written (14237)(56).

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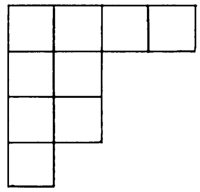
(1.1) Suppose that $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_s)$ is a shape. A *tableau* of shape λ consists of s top-justified columns of respective sizes $\lambda_1, \ldots, \lambda_s$. Thus $(4 \geqslant 3 \geqslant 1 \geqslant 1)$ corresponds to



A *standard Young tableau* of shape λ is a tableau filled with the integers $1, \ldots, \Sigma\lambda_i$ such that the rows decrease from left to right and the columns decrease from top to bottom (this is not the usual convention). Thus, the standard Young tableaux of shape $3 \geqslant 2$ are

5	4
3	2
1	

5	4
3	1
2	

5	3
4	2
1	

5	3
4	1
2	

5	2
4	1
3	

(1.2) There is a one-to-one correspondence between pairs of Young tableaux with n boxes and elements of S_n, the symmetric group; this is the *Robinson-Schensted correspondence* (see, e.g. **[Kn]**). We use the following nonstandard insertion algorithm: let $t_1, \ldots, t_s$ be the top row of a tableau T, and let T' be the remainder of T. We insert l as follows: if $l > t_1$, the resulting tableau has top row $l, t_2, \ldots, t_s$, and t_1 is inserted into T'. If $t_i > l > t_{i+1}$, the top row of the resulting tableau is $t_1, \ldots, t_i, l, t_{i+2}, \ldots, t_s$ and t_{i+1} is inserted into T'. (It is convenient to assume $-\infty$ is at the end of each row.) To create a pair of tableaux corresponding to $\sigma \in S_n$, insert $\sigma(n), \sigma(n-1), \ldots, \sigma(1)$ in the left tableau while filling the right tableau with $n, n-1, \ldots, 1$ as each new cell is created. To construct a permutation from a tableau pair, exchange the left and right tableaux and imagine how each element was inserted, starting with the right tableau entry whose place is occupied by 1 in the left tableau.

(1.3) EXAMPLE. We construct the tableau pair for $\alpha\alpha\beta\alpha\beta$. The permutation σ has the following action: $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 5, \sigma(4) = 1, \sigma(5) = 4$. We insert 4, 1, 5, 2 and 3 into a tableau, with the result:

5	3
4	2
1	

5	4
3	2
1	

Conversely, take the pair

$$\begin{matrix} 5 & 4 \\ 3 & 2 \\ 1 & \end{matrix} \qquad \begin{matrix} 5 & 3 \\ 4 & 2 \\ 1 & \end{matrix}$$

The 1 on the right was "bumped" by 2, which was bumped by 3, so $\sigma(1) = 3$, etc.

(1.4) PROPOSITION. *The tableaux associated to $w \in M_a^b$ have 2 columns, the second of which has length $\leqslant b$.*

PROOF. Notice that the longest increasing subsequence in the permutation associated to w has length 2. The result now follows from [**Kn**, 5.1.4, exercise 7]. □

(1.5) Let $u \in \mathrm{SL}_n$ be a unipotent of shape λ. Following [**S**], we set up a one-to-one correspondence between components of $\mathscr{B}_u$ and Young tableaux of shape λ. Let e be the $n \times n$ identity, and write $\tilde{u} = u - e$. Suppose T is a tableaux of shape λ such that 1 is in row ρ. Then the generic flag $F_1 \subset F_2 \subset \cdots \subset F_n$ in the component associated to T satisfies

$$F_1 \subset \ker(\tilde{u}) \cap \left[\operatorname{image}(\tilde{u}^{\rho-1}) \setminus \operatorname{image}(\tilde{u}^{\rho})\right];$$

$F_2, \ldots, F_n$ are determined by following this procedure inductively.

(1.6) It is now clear how $w \in M_a^b$ corresponds to a component of $\mathscr{B}_u$, where u has shape $\lambda_1 \geqslant \lambda_2$ with $\lambda_1 \geqslant a$, $\lambda_2 \leqslant b$. Assume that the first letter of w is α (if not, the second columns of the tableaux associated to w have length $< b$, since n is the last integer inserted); then w corresponds to a pair of tableaux (T_L, T_R) of shape $a \geqslant b$. Take the tableau T_R; it corresponds to a component of $\mathscr{B}_u$. Conversely, a component of $\mathscr{B}_u$ corresponds to a tableau T_R, and (T_L, T_R) corresponds to some $w \in M_a^b$ (that this is unique follows from the proof of (1.4)).

(1.7) We *label* each $w \in M_a^b$ with the integer j if the jth, $(j+1)$st letters of w are $\alpha\beta$; thus $\alpha\alpha\beta\alpha\beta$ has the labels 2 and 4.

REMARK. The labels of w are clearly the left descent set of w when S_n is thought of as a Coxeter group of type A_{n-1}.

PROPOSITION. *The labels of w are the parabolic lines contained in the component of $\mathscr{B}_u$ associated to w.*

PROOF. Recall that a component of $\mathscr{B}_u$ contains parabolic lines of type j if one is free to choose F_j such that $F_{j-1} \subset F_j \subset F_{j+1}$ when F_{j-1}, F_{j+1} are fixed in $F_1 \subset \cdots \subset F_n$. It is clear from (1.5) that this is the case exactly when j is above or at the same level as $j+1$ in the tableau associated to the component.

Suppose w has the label j, and that σ is the associated permutation. Then $\sigma(j+1) > \sigma(k)$ for $k > j+1$, since there is a β in the $(j+1)$st spot and β's increase from right to left. Thus, the new cell in the tableau is at the bottom of the first column, since $\sigma(j+1)$ "bumps" the element in the upper left-hand corner. However, $\sigma(j) < \sigma(j+1)$, since all α are less than all β, so the new cell for $\sigma(j)$ is at the end of column 2. Since column 2 is always shorter than column 1 (or equal in length), the result follows. □

This can often be used to associate a word to a tableau quickly, since e.g. the tableau

$$\begin{matrix} 5 & 4 \\ 3 & 2 \\ 1 & \end{matrix}$$

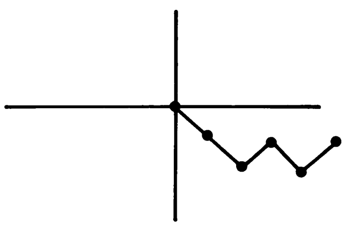
indicates lines of type 2 and 4, hence labels 2 and 4, hence $\alpha\alpha\beta\alpha\beta$.

(1.8) PROPOSITION. *Suppose $w \in M_a^b$ begins with $\alpha\beta\beta$. Then w does not correspond to a component of $\mathscr{B}_u$ where u has shape $a \geqslant b$.*

PROOF. Suppose the contrary; then the second column of the associated tableau has length b. Write $w = \alpha\beta\beta w'$, where $w' \in M_{a-1}^{b-2}$; the second columns of the w' tableaux have length $\leqslant b - 2$. Insertion of $a + b - 1$, $a + b$, and a into these tableaux give the w tableaux, but the first 2 elements go into the first column, so the second column has length $\leqslant b - 1$. This is a contradiction. $\square$

2.0. A pair $(v, w) \in M_a^b \times M_a^b$ is *reduced* if all coincident phrases $\alpha\beta$ are removed, e.g. if $(v, w) = (\alpha\underline{\alpha\beta}\alpha\beta, \alpha\underline{\alpha\beta}\beta\alpha)$, the reduced pair is $(\alpha\alpha\beta, \alpha\beta\alpha)$. This has the effect of removing common labels.

Each $w \in M_a^b$ is associated to a path in the plane $\mathbf{R}^2$ by starting at $(0,0)$ and assigning segments $(m, n) \to (m + 1, n - 1)$ to each α and $(m, n) \to (m + 1, n + 1)$ to each β, e.g. $\alpha\alpha\beta\alpha\beta$:



We can now state our result.

(2.1) THEOREM. *Associate $v, w \in M_a^b$ to components X_v, $X_w \subset \mathscr{B}_u$, where u has shape $a \geqslant b$.*

(a) *Suppose the reduced form of (v, w) is*

$$v^{\mathrm{red}} = w_0 x_1 w_1 x_2 w_2 \cdots x_m w_m, \qquad w^{\mathrm{red}} = w_o \tilde{x}_1 w_1 \tilde{x}_2 w_2 \cdots \tilde{x}_m w_m,$$

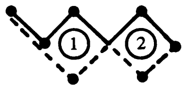
where $x_i \in M_1^1$ and $\widetilde{\alpha\beta} = \beta\alpha$, $\widetilde{\beta\alpha} = \alpha\beta$. Then

$$\operatorname{codim}_{X_v}(X_v \cap X_w) = m.$$

(b) *If v^{red} and w^{red} do not have this form, then the components do not meet.*

REMARK. The codimension is the number of "diamonds" when the paths of (2.0) are superimposed, e.g.,

Superimpose:



Codimension is 2.

3.0.

(3.1) We prove (2.1) by induction on b. When $b = 1$, the components have dimension 1, and may be identified by their labels, as each is a parabolic line. Parabolic lines of type i meet those of type j when $j = i + 1$ or $i = j + 1$. The words are

$$\alpha \cdots \alpha\beta\alpha \cdots \alpha \qquad (\beta \text{ in } (i+1)\text{st spot}),$$
$$\alpha \cdots \alpha\alpha\beta\alpha \cdots \alpha \qquad (\beta \text{ in } (i+2)\text{nd spot}).$$

The conclusion follows.

This also proves (2.1)(b).

(3.2) Notice that the removal of a label reduces the component dimension by 1 (by (1.8), no word begins $\alpha\beta\beta$). If components X_1, X_2 both contain parabolic lines of type j, and if $\mathscr{F} = (F_1 \subset F_2 \subset \cdots \subset F_{j-1} \subset F_j \subset F_{j+1} \subset \cdots \subset F_n)$ is a flag in $X_1 \cap X_2$, then so is $F_1 \subset \cdots \subset F_{j-1} \subset F_j' \subset F_{j+1} \subset \cdots \subset F_n$. Thus, the existence of a common label adds 1 to the dimension of each component and 1 to the dimension of the intersection, so the codimension is unchanged. Thus, we can assume that v and w are reduced.

(3.3) Suppose that

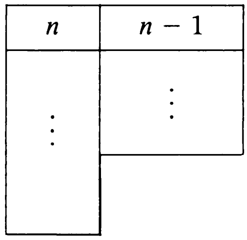
$$v = v_0 x_1 v_1 \cdots x_m v_m, \qquad w = v_0 \tilde{x}_1 v_1 \cdots \tilde{x}_m v_m.$$

The word v_m has the form $\beta \cdots \beta\alpha \cdots \alpha$, since it has no label. Suppose it has p β's and q α's; then by (1.7), the subspaces $F_{n-p-q} \subset \cdots \subset F_n$ of a general flag in each component are forced, and we can reduce to the case of a unipotent of shape $a - p \leqslant b - q$ acting on F_{n-p-q}; the result is known here by induction.

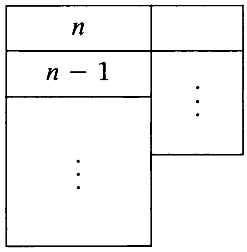
Thus, renaming if necessary, we are reduced to the case

$$v = v_0 x_1 v_1 \cdots v_{m-1}\alpha\beta, \qquad w = v_0 \tilde{x}_1 v_1 \cdots v_{n-1}\beta\alpha.$$

The tableau describing X_v is



and that for X_w is



Choose $\mathscr{F} = (F_1 \subset \cdots \subset F_n)$ in the intersection. F_n is the space on which u acts, while F_{n-1} is forced by the tableau for w. However, there is a $\mathbf{P}^1$ of choices for F_{n-1} in the component for v, since it contains parabolic lines of type $n-1$. Thus, we lose 1 dimension of choice in X_w, so

$$\operatorname{codim}_{X_w}(X_v \cap X_w) = 1 + \operatorname{codim}_{X_{\tilde{w}}}(X_{\tilde{v}} \cap X_{\tilde{w}}),$$

where

$$\tilde{v} = v_0 x_1 v_1 \cdots v_{m-1}, \qquad \tilde{w} = v_0 \tilde{x}_1 v_1 \cdots v_{m-1}.$$

This induction step completes the proof. □

4.0. Here is an application of (2.1). Let $l\colon S_n \to \mathbf{Z}$ be the length function. For each $y, w \in S_n$, there is a *Kazhdan-Lusztig polynomial* $P_{y,w} \in \mathbf{Z}[q]$ whose degree does not exceed $\frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ in the Bruhat order. Let $\mu(y, w)$ be the coefficient of the term of that degree in $P_{y,w}(q)$, and, following [**KL**, §1], define an "*S_n-graph*" as follows: the nodes are elements of S_n, and elements y and w are connected by an edge when either $\mu(y, w)$ or $\mu(w, y)$ is nonzero. Further, each node is labeled with its left descent set.

(4.1) Define another family of graphs as follows: let $u \in \mathrm{SL}_n$ be a unipotent. The nodes are components of $\mathscr{B}_u$, and two such are connected if the components meet in codimension 1. The nodes are labelled with the set of parabolic lines contained in the associated component.

(4.2) Suppose that $y_1, y_2 \in S_n$ are permutations constructed, as in (1.0), from w_1, w_2.

THEOREM [**LS**]. *Suppose the reduced forms of w_i are*

$$w_1 = v_0 x_1 v_1 \cdots x_m v_m, \qquad w_2 = v_0 \tilde{x}_1 v_1 \cdots \tilde{x}_m v_m,$$

with $x_i \in M_1^1$. Then $\mu(y_1, y_2) = 1$ when $m = 1$; otherwise $\mu(y_1, y_2) = 0$.

(4.3) In [**KL**, 6.3] it is suggested that each graph in (4.1) should be a "W-graph", as in (4.0). It is an immediate consequence of (2.1) and (4.2) that this is the case when u is a unipotent with shape $a \geqslant b$. In particular, the intersection graph of $\mathscr{B}_u$ will be exactly the "left cell" containing the involution corresponding to $w = \alpha \cdots \alpha\beta \cdots \beta$, where w has a α's and b β's (this is an involution since it is a product of the longest elements of S_a acting on $\{1, \ldots, a\}$ and S_b acting on $\{a + 1, \ldots, a + b\}$). See (1.7) to verify that the labelling of the nodes is correct.

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