

ON FREE SUBSEMIGROUPS OF SKEW FIELDS

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ABSTRACT. I will prove in this note that the multiplicative group of a skew field with uncountable center either contains a free nonabelian subsemigroup or is commutative.

Let T be a skew field with center Z and let us suppose that Z is an uncountable set. Then the following statement is true.

THEOREM. *If $T \neq Z$ then the multiplicative group of T contains a free subsemigroup with two generators.*

The proof of the theorem is broken into the following three lemmas.

LEMMA 1. *Suppose T has no such subsemigroup. Then for any pair of elements x, y from T there exists a semigroup relation $A(a, b) = B(a, b)$ such that this relation is valid for every pair $x + c, y + dx$, where $c, d \in Z$, and that $\deg_a A = \deg_a B$, $\deg_a A = \deg_b B$.*

PROOF. Let us fix a pair x, y and consider all pairs $x + c, y + dx$. Each pair fails to span a free semigroup (with two generators) so it satisfies some relation which of course depends on c and d . Now we have only a countable number of possible relations. This means that if we fix, for example, $c = c_0$ then there exists a relation which is satisfied by an infinite number of pairs $x + c_0, y + dx$. Let us represent this relation as a polynomial in d with coefficients depending on x, y and c_0 . This polynomial has infinitely many roots so all of its coefficients must be zeros (we can use Vandermond's determinant with entries $\{d_i^j\}$). So this relation is satisfied by the pair $x + c_0, y + dx$ for every d in Z . Thus for every c there exists a relation $P(c)$ of this type which is satisfied by every pair $x + c, y + dx$ (where c is fixed and d can be taken arbitrarily). Let us call such relations *good* for x, y and c . Among the countably many good relations we can find a relation P which is satisfied by $x + c, y + dx$ for infinitely many c (and any d). Then the same reasoning shows that this relation P is satisfied by $x + c, y + dx$ for every c, d (because for every fixed d this relation is satisfied for infinitely many c and thus for every c). Let us call such relations *better* for x, y . Now for each pair ex, fy , where $e, f \in Z$, there exists a better relation for ex, fy and again we can see that there exists a better relation which is satisfied by every pair ex, fy . By factoring out e and f we can see that this relation has the same degrees in a and in b on its left and right sides.

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LEMMA 2. *If the relation $A(a, b) = B(a, b)$ is nontrivial and satisfies the conclusion of Lemma 1 for a pair $x, y \in T$ then there exists some $n = n(x, y)$ such that $(ad_x)^n y = 0$, where $ad_x y = [x, y] = xy - yx$.*

PROOF. We can write $A(a, b + da)$ as $\sum_j A_j(a, b)d^{J-j}$, where $\deg_b A_j = j$. Let us show first that $A_1(a, b)$ uniquely defines $A(a, b)$ if $\deg_b A \neq 0$. Suppose that $A(a, b) = a^{i_1} b^{j_1} \dots a^{i_k} b^{j_k}$. Then $A_1(a, b) = \sum_{m \in M} a^m b a^{I+J-m-1}$, where $I = \sum i_s, J = \sum j_s$. The set M can be represented as the union of the longest possible arithmetic progressions with difference equal to one. The first of these progressions starts with i_1 and contains j_1 terms, the second one starts with $i_1 + j_1 + i_2$ and contains j_2 terms, the m th progression starts with $i_1 + j_1 + \dots + j_{m-1} + i_m$ and contains j_m terms. This information, together with the known $\sum(i_s + j_s)$, uniquely defines all i 's and j 's.

Now we may assume that both $\deg_a A$ and $\deg_b A$ are nonzero because otherwise the relation is trivial. So $A_1 = B_1$ is a nontrivial relation. We can rewrite this relation as the equivalent relation

$$(*) \quad \sum_{s=0}^{I+J} k_s a^s b a^{I+J-s} = 0.$$

(Here all the k_s are 0 or ± 1 but we are going to use only that $k_s \in Z$.) We can rewrite relation (*) as

$$\sum_{s=0}^{I+J-1} k'_s a^s b_1 a^{I+J-1-s} + k'_{I+J} a^{I+J} b = 0,$$

where $b_1 = [b, a]$. Now $k'_{I+J} c^{I+J} y = 0$ which means that $k'_{I+J} = 0$. So instead of the relation (*) we obtain the relation

$$(**) \quad \sum_s k'_s a^s b a^{I+J-1-s} = 0$$

of degree smaller than the degree of relation (*) and which is satisfied by every pair $a = x + c, b = [y, x]$. Now we may apply induction on the degree \deg_a of the relation. The inductive hypothesis is that from every relation of type (*) which is satisfied by all pairs $a = u + c, b = v$, where u and v are any fixed elements of T and c is a nonfixed element of Z , it follows that $(ad_u)^n v = 0$ for some n . The base of induction, when this degree is one, is trivial. For the induction step we remark that if we start with a nontrivial relation then we obtain a nontrivial relation again because in (*) and (**) the first nonzero coefficients k_s and k'_s have the same subscript and are equal to each other. This finishes the proof of this lemma.

Now if we assume that T does not contain a free subsemigroup with two generators, we can deduce with the help of Lemmas 1 and 2 that for any pair $x, y \in T$ there exists $n = n(x, y)$ such that $(ad_x)^n y = 0$. If T is not a field, let us take a pair x, y which does not commute. Let us denote $(ad_x)^{n-2} y ((ad_x)^{n-1} y)^{-1}$ by w (here $n > 1$ is the minimal power of ad_x which annihilates y). It is clear that $[x, w] = 1$. Let us denote the skew subfield spanned by x and w over Z by D . The proof of the theorem will be finished if we prove the following lemma.

LEMMA 3. *The skew field D contains a free semigroup with two generators.*

PROOF. If D does not contain such a semigroup, then by Lemma 2 for the elements $t = xw$ and w there exists an n such that $(ad_t)^n w = 0$. But $[t, w] = w$ so $(ad_t)^n w = w$ for any n . This contradiction shows that D contains a desired subgroup and so does T .

REMARK 1. It is possible to prove that every skew field which is finite dimensional over its center and does not coincide with its center contains a noncommutative free subgroup. (This result can be extracted from the papers of A. Lichtman [1 and 2], see also [3].) So the following conjecture sounds quite reasonable: every skew field which is really skew contains a noncommutative free subsemigroup.

REMARK 2. The theorem stated can be looked at as a generalization of a result of I. Kaplansky [4] although the restriction on the cardinality of the center somewhat spoils the picture.

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