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SEPARABLE ABELIAN GROUPS AS MODULES OVER THEIR ENDOMORPHISM RINGS

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ABSTRACT. Properly separable mixed abelian groups A which are projective, respectively flat, as modules over their endomorphism rings are completely characterized. These results generalize the works of F. Richman and E. A. Walker.

A subclass of separable mixed abelian groups—the properly separable groups—in which finitely many rank one summands can be embedded in a finite rank summand is considered here. It is shown that a reduced properly separable mixed abelian group A is projective as a module over its endomorphism ring $E(A)$ exactly when $A = T \oplus (\bigoplus B_i)$, $i \in I$, where T is torsion with each of its p-components bounded, for each $i \in I$, B_i contains a torsionfree summand of rank one whose type is idempotent and is the smallest in the typeset of $B_i/(B_i)_t$ and, finally, the subgroups T and B_i are all fully invariant in A. A result of F. Richman and E. A. Walker is generalised to show that the reduced properly separable mixed abelian groups A which are $E(A)$-flat have the following characterising property: In the typeset of A/A_t, any nonempty finite subset has a lower bound whenever it has an upper bound.

All the groups that we consider here are additively written abelian groups. For the general notation and terminology, we refer to [**3**]. By a group of rank one we mean a subgroup of rational numbers or a subgroup of the Prüfer group $Z(p^\infty)$ for some prime p. A mixed group A is said to be *separable* if each nonempty finite subset can be embedded in a direct summand of A which is a direct sum of finitely many rank one groups. For the general properties of separable mixed groups, we refer to [**1** and **4**]. As noted in [**4**], the height matrices of elements in a reduced separable mixed group have finitely many jumps and have as entries the nonnegative integers and the symbol ∞. We shall call such heights matrices separable. For two separable height matrices H and K, define $H \sim K$, if (i) $H = K$, in case almost all the entries of H are ∞, and (ii) $mH = nK$ for some positive integers m, n, in case H has infinitely many finite entries. Then $\sim$ is an equivalence relation and each equivalence class is called a *type*. For two types t_1 and t_2, we define $t_1 \geq t_2$ if there are height matrices $H_1 \in t_1$ and $H_2 \in t_2$ with $H_1 \geq H_2$. Thus $t_1 \geq t_2$ if t_1 is the height matrix type of an element of order p^n and t_2 is that of a torsionfree element with finite p-height in A. For any reduced separable group A, let $\Gamma(A)$ denote the set of types of all its rank one summands.

A separable group A is said to be *properly separable* if finitely many rank one

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summands can be embedded in a finite rank summand. Clearly, a separable group which is torsion or torsionfree is properly separable. As indicated in [**4**], if A is properly separable and nontorison, then for each prime p relevant to A, there is a rank one torsionfree summand whose type has 0 or ∞ as the pth entry. We need the following result proved in [**4**]:

LEMMA 1. *A properly separable reduced group A decomposes as $A = T \oplus B$, where T is torsion, B is nontorsion with the property that, for each prime p relevant to B, there is a $t \in \Gamma(B/B_t)$ whose pth entry is 0 and both T and B are fully invariant in A.*

For later use, we say that a group G has the *Property* $(*)$ if it has the property of the group B stated in Lemma 1.

Our next theorem characterizes the properly separable groups A which are flat over their endomorphism rings $E(A)$. F. Richman and E. A. Walker [**6**] described the torsionfree completely decomposable groups A which are $E(A)$-flat. Our proof originated from theirs and the author is grateful to them for having given access to their preprint [**6**].

THEOREM 2. *Let A be a reduced properly separable abelian group and let $E = E(A)$. Then the following are equivalent:*

(i) *A is E-flat.*

(ii) *For any two rank one summands M_1, M_2, whenever $\mathrm{Hom}(M_1, M_3) \neq 0$ and $\mathrm{Hom}(M_2, M_3) \neq 0$ for any rank one summand M_3, then there is a rank one summand M_4 such that $\mathrm{Hom}(M_4, M_1) \neq 0$ and $\mathrm{Hom}(M_4, M_2) \neq 0$.*

(iii) *In $\Gamma(A)$ any nonempty finite subset has a lower bound whenever it has an upper bound.*

PROOF. (i)$\Rightarrow$(ii). Let $0 \neq f_1: M_1 \to M_3$ and $0 \neq f_2: M_2 \to M_3$. Then $f_1(x_1) = f_2(x_2) \neq 0$ for some $x_i \in M_i$, $i = 1, 2$. By the flatness of A, there exist $y_j \in A$, $f_{ij} \in E$, for $i = 1, 2$ and $j = 1, \ldots, t$, such that $x_i = \sum_{j=1}^{t} f_{ij} y_j$, for $i = 1, 2$ and $f_1 f_{1j} = f_2 f_{2j}$, for all $j = 1, \ldots, t$ (see [**2**]). Then $f_1 f_{1j}(y_j) = f_2 f_{2j}(y_j) \neq 0$ for some j. Since A is separable, $f_1 f_{1j}(z) = f_2 f_{2j}(z) \neq 0$ for some z in a rank one summand M_4 of A. Then $\mathrm{Hom}(M_4, M_1) \neq 0 \neq \mathrm{Hom}(M_4, M_2)$.

(ii)$\Rightarrow$(iii). Obvious.

(iii)$\Rightarrow$(i). We first show that $A[\tau]$ is a flat E-module if τ is the type of a torsionfree summand S of rank one in A. Let $\tau = (k_1, \ldots, k_n, \ldots)$. Note that $k_i = 0$ or ∞ if the associated prime p_i is relevant to A (because A is properly separable). Let $x_0 \in S$ such that x_0 has characteristic $(k_1, \ldots, k_n, \ldots)$. Let $K = \{k_i: k_i \neq 0 \text{ or } \infty\}$, $\{r_1, r_2, \ldots\}$ an enumeration of K, and $q_1, q_2, \ldots$ the corresponding primes. Define, for each i, $x_i \in S$ such that $q_1^{r_1} q_2^{r_2} \cdots q_i^{r_i} x_i = x_0$. Now, for any prime p relevant to A, each x_i has p-height 0 or ∞, so that if, for any $f \in E$, $f(x_i) = 0$, then $f(S) = 0$. If $e: A \to S$ is a projection, it then follows that for each i, $Ee \simeq Ex_i$ as an E-module under the map $fe \mapsto f(x_i)$. Since $A[\tau]$ is the union of the ascending chain of projective E-modules $Ex_0 \subset Ex_1 \subset Ex_2 \subset \cdots$, $A[\tau]$ is a flat E-module.

Now by Lemma 1, $A = T \oplus B$, with T torsion separable, $\oplus$ an E-module direct sum and B having property $(*)$. By [**5**], T is $E(T)$-flat so we shall prove the result assuming that $A = B$. Let $\Gamma = \Gamma(A/A_t)$. For any $\tau_1, \tau_2 \in \Gamma$, define $\tau_1 \sim \tau_2$ if there is a $\tau \in \Gamma$ such that $\tau \leq \tau_1, \tau_2$. By hypothesis, $\sim$ is an equivalence relation. Let

$\{\xi_m : m \in M\}$ be a partitioning of Γ induced by $\sim$. Let, for each $m \in M$, $A_m = \bigcup\{A[\tau] : \tau \in \xi_m\}$. Clearly $\mathrm{Hom}(A_m, A_n) = 0$, for any $m, n \in M$ with $m \neq n$. The properties of A then imply that A is a direct sum of the A_m's. The result follows if one notes that each A_m, being a directed union of the flat E-modules $\{A[\tau]\}, \tau \in \xi_m$, is itself E-flat.

REMARK. From the first part of the proof of (iii)⇒(i), we note the following: If A is an arbitrary torsionfree abelian group and τ is the type of a rank one summand of A, then $A[\tau]$ is $E(A)$-flat.

We shall now consider the properly separable groups which are projective over their endomorphism rings. For any type τ, let τ_0 denote the reduced type of τ obtained by replacing the finite entries of τ by 0. The *reduced inner type* σ of a separable group A is defined by $\sigma = \min\{\tau_0 : \tau \in \Gamma(A)\}$.

We begin with the following useful lemma.

LEMMA 3. *Suppose A is a reduced nontorsion properly separable group. Suppose any two τ_1, τ_2 in $\Gamma(A)$ have a lower bound in $\Gamma(A)$. Then the center of $E(A)$ is a subring R_0 of rational numbers whose type is the reduced inner type of A.*

PROOF. Clearly the multiplication by any rational number $r \in R_0$ acts as an endomorphism of A and is in the center of $E(A)$. Conversely, let f be an element of the center of $E(A)$. Then f acts as an endomorphism of each rank one summand of A and hence as a multiplication by a rational number r whose type has entries 0 at those primes p at which the type of some rank one summand of A has a finite entry. Hence $r \in R_0$.

We are now ready to characterize those separable mixed groups A which are $E(A)$-projective. These turn out to be direct sums of (cyclic) summands of $E(A)$.

THEOREM 4. *Let A be a reduced properly separable mixed abelian group and let $E = E(A)$. Then A is projective as an E-module if and only if $A = T \oplus (\bigoplus B_i)$, $i \in I$, where T and the B_i are fully invariant, T is torsion with each of its p-components bounded and, for each $i \in I$, B_i contains a rank one torsionfree summand whose type is the reduced inner type of B_i.*

PROOF. Suppose A is E-projective. By Lemma 1, we have an E-module decomposition $A = T \oplus B$, with T torsion and B having property $(*)$. Since T is torsion and $E(T)$-projective, T_p is bounded for each p [**5**]. Since B is, in particular, $E(B)$-flat, it follows from the proof of Theorem 2, that $B = \bigoplus B_i$, $i \in I$, where $\bigoplus$ is an E-module direct sum and, for each $i \in I$, $\Gamma(B_i)$ is directed below. For convenience in writing, let us replace a nontorsion B_i by A. By the assumption on B_i, A is a directed union of $\{A[\tau] : \tau \in \Gamma(A)\}$. Since A is E-projective, there exists, by the dual basis theorem [**2**], $f_j \in \mathrm{Hom}_E(A, E)$, $x_j \in A$ with $j \in J$, such that, for any $x \in A$, $f_j(x) = 0$ for almost all $j \in J$ and $x = \sum f_j(x)x_j$. Then the Z-morphism

$$\alpha : \mathrm{Hom}_E(A, E) \otimes_E A \to \mathrm{Hom}_E(A, A)$$

given by $\alpha(f \otimes a)(x) = f(x)a$ is nonzero since $\alpha(\sum(f_j \otimes x_j))(x) = \sum f_j(x)x_j = x$. As A is a directed union of $A[\tau]$, $\tau \in \Gamma(A)$, $\alpha(\mathrm{Hom}_E(A, E) \otimes A[\tau]) \neq 0$ for some $\tau \in \Gamma(A)$. By Lemma 3, $\mathrm{Hom}_E(A, A) = R_0$, the subring of rational numbers containing 1 and having the reduced inner type σ_0 of A. It is then readily seen that

$\tau \leq \sigma_0$. Thus A has a rank one torsionfree summand whose type is the reduced inner type of A.

Conversely, let $A = T \oplus (\bigoplus B_i)$, $i \in I$, with T torsion having all its p-components bounded and the B_i having the stated properties. Let $\gamma_i: A \to B_i$, $i \in I$, be the projections corresponding to the given decomposition. If each B_i contains a rank one summand S_i with $e_i = e_i\gamma_i: A \to S_i$ a corresponding projection and S_i having an element x_i whose height matrix = the reduced inner type of B_i, then the map $fe_i \mapsto f(x_i)$ is an E-isomorphism of Ee_i with B_i. Thus each B_i is E-projective. Since, by [**5**], T is E-projective, so is A.

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