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FRÉCHET DIFFERENTIATION OF CONVEX FUNCTIONS IN A BANACH SPACE WITH A SEPARABLE DUAL

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ABSTRACT. Let X be a real Banach space with a separable dual and let f be a continuous convex function on X. We sharpen the well-known result that the set of points at which f is not Fréchet differentiable is a first category set by showing that it is even σ-porous. On the other hand, a simple example shows that this set need not be a null set for any given Radon measure.

Let X be an Asplund space. By definition, this means that if f is a continuous convex function on X, then the set of all points at which f is not Fréchet differentiable is a first category set. The natural question arises (cf. [**1**]) of finding the most strict sense in which this set is small. In Theorem 1 we give a partial answer to this question in case of a space with separable dual by showing that the set of points of Fréchet nondifferentiability of any continuous convex function on such a space is σ-porous. It might be worth noting that the proof of Theorem 1 is, as far as we know, the simplest proof of the result of Asplund that every space with separable dual is an Asplund space. For this purpose, one does not even need the notion of σ-porosity, since the corresponding part of the proof of Theorem 1 can be read as a proof that each of the sets $A_{m,k}$ is nowhere dense.

Let P be a metric space. The open ball with the center $x \in P$ and the radius $r > 0$ is denoted by $B(x,r)$. Let $M \subset P$, $x \in P$, $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $z \in P$ such that $B(z,r) \subset B(x,R) - M$ by $\gamma(x,R,M)$. The number $\limsup_{R\to 0_+} \gamma(x,R,M)R^{-1}$ is called the porosity of M at x. If the porosity of M at x is positive we say that M is porous at x. A set is said to be porous if it is porous at all its points. A set is termed σ-porous if it can be written as a union of countably many porous sets.

It is easy to see that any porous set is nowhere dense and therefore any σ-porous set is a first category set. Since a σ-porous subset of R can contain no density points, it must be of Lebesgue measure zero. Using this fact, one easily notes (cf. proof of Lemma 3.4 in [**2**]) that, whenever X is a Banach space, $p \in X^*$, $p \neq 0$, and $K \subseteq R$ is a nowhere dense set of positive Lebesgue measure, then $p^{-1}(K)$ gives an example of a first category set in X which is not σ-porous. Therefore the following theorem contains a new result.

THEOREM 1. *Let X be a real Banach space with a separable dual and let f be a continuous convex function on X. Then the set A of points at which f is not Fréchet differentiable is σ-porous.*

PROOF. For any $x \in A$ we choose a $p^x \in X^*$ such that $f(x+h) - f(x) \geq (h, p^x)$

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for every $h \in X$ (i.e., $p^x \in \partial f(x)$) and we find a natural number m_x such that

$$\limsup_{h\to 0}(f(x+h) - f(x) - (h, p^x))\|h\|^{-1} > \frac{1}{m_x}. \tag{1}$$

Put $A_m = \{x \in A; m_x = m\}$. Since X^* is separable we can choose for each m a sequence $(A_{m,k})_{k=1}^{\infty}$ such that $A_m = \bigcup_{k=1}^{\infty} A_{m,k}$ and

$$\|p^x - p^y\| < 1/6m \quad \text{whenever } x, y \in A_{m,k}. \tag{2}$$

Obviously $A = \bigcup_{m,k=1}^{\infty} A_{m,k}$ and therefore it is sufficient to show that each of the sets $A_{m,k}$ is porous.

Let m, k and $x \in A_{m,k}$ be fixed. Since any continuous convex function is locally Lipschitz we can choose $R > 0$, $K > 1/m$ such that the Lipschitz constant of f on $B(x, R)$ does not exceed K. It is easy to see that $\|p^x\| \le K$. By (1), for every $r > 0$ there exists $h \in X$ such that

$$f(x+h) - f(x) > \|h\|/m + (h, p^x) \quad \text{and} \quad \|h\| < \min(r, R/2). \tag{3}$$

To prove that $A_{m,k}$ is porous at x it is sufficient to show that

$$B(x+h, \|h\|/3Km) \cap A_{m,k} = \emptyset. \tag{4}$$

Suppose on the contrary that there is some $y \in B(x+h, \|h\|/3Km) \cap A_{m,k}$. By the definition of p^y we have

$$f(x) - f(y) \ge (x - y, p^y). \tag{5}$$

Using (3) and (5) we obtain

$$\begin{aligned} f(x+h) - f(y) &> (h, p^x) + (x - y, p^y) + \|h\|/m \\ &= ((x+h) - y, p^x) + (x - y, p^y - p^x) + \|h\|/m. \end{aligned}$$

Since $\|(x+h) - y\| < \|h\|/3Km$ and $\|x - y\| < 2\|h\|$ we obtain $|((x+h) - y, p^x)| < \|h\|/3m$, $|(x - y, p^y - p^x)| < \|h\|/3m$ and, consequently,

$$f(x+h) - f(y) < \|h\|/3m. \tag{6}$$

On the other hand, $x + h \in B(x, R)$, $y \in B(x, R)$ and therefore $f(x+h) - f(y) \le K\|(x+h) - y\| < \|h\|/3m$, which contradicts (6). Thus (4) is proved and the proof is completed.

THEOREM 2. *Let X be a Banach space and $\emptyset \ne C \subset X$ a closed convex set with empty interior. Then the distance function $d(x) = \inf\{\|x - y\|; y \in C\}$ is Fréchet differentiable at no point of C.*

PROOF. It is easy to see that any distance function determined by an arbitrary nonempty set $M \subset X$ is Fréchet differentiable at $x \in M$ iff M is not porous at x, the derivative being in this case the zero functional.

Let $x \in C$ and $r > 0$ be fixed. Choose $y \in B(x, r/3) - C$. By the well-known separation theorem we can choose $p \in X^*$, $\|p\| = 1$, such that

$$(c, p) < (y, p) \quad \text{for any } c \in C. \tag{7}$$

Choose $v \in X$, $\|v\| = 1$, such that $(v, p) > 1/2$, and put $z = y + 3^{-1}rv$. To prove that C is porous at x it is sufficient to prove that $B(z, r/6) \cap C = \emptyset$. Suppose on the contrary that there exists $t \in B(z, r/6) \cap C$. Then by (7), $(t-y,p) < 0$. On the other hand,

$$(t-y,p) = ((z-y)+(t-z),p) > (3^{-1}rv,p) - |(t-z,p)| > r/6 - r/6 = 0,$$

which is a contradiction.

COROLLARY. *Let μ be a Radon measure on an infinite-dimensional Banach space X. Then there exists a continuous convex function f on X and a set $A \subset X$, $\mu A > 0$, such that f is Fréchet differentiable at no points of A.*

PROOF. Since μ is a Radon measure there exists a compact set $K \subset X$ such that $\mu K > 0$. Let A be the closed convex hull of K and let f be the distance function determined by A. Then f is a continuous convex function and, since A is compact, f is Fréchet differentiable at no point of A according to Theorem 2.

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