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WEIGHTED NORM INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON SPACES OF HOMOGENEOUS TYPE

HUGO AIMAR AND ROBERTO A. MACÍAS

ABSTRACT. The purpose of this note is to give an adequate Calderón-Zygmund type lemma in order to extend to the general setting of spaces of homogeneous type the A_p weighted L^p boundedness for the Hardy-Littlewood maximal operator given by M. Christ and R. Fefferman.

Recently Michael Christ and Robert Fefferman gave in [1] a remarkable proof of the weighted norm inequality for the Hardy-Littlewood maximal function operator in $\mathbf{R}^n$, $\|Mf\|_{L^p(w)} \leqslant C_p\|f\|_{L^p(w)}$ when the weight belongs to Muckenhoupt A_p classes and $p > 1$. In [2], A. P. Calderón proved this boundedness property for spaces such that the measure of balls is continuous as a function of the radius. In [3], R. Macías and C. Segovia extended this result to general spaces of homogeneous type (defined below) constructing an adequate quasi-distance. In both cases, the reverse Hölder inequality must be extended to this general setting, while the proof given in [1] does not make use of this property and only depends on an adequate Calderón-Zygmund type lemma, the proof of which for cubes in $\mathbf{R}^n$ is very simple. The purpose of this note is to obtain a decomposition lemma which allows us to extend the proof of Christ and Fefferman to spaces of homogeneous type.

We now introduce some notation and definitions. Let X be a set, a nonnegative symmetric function on $X \times X$ shall be called a quasi-distance if there exists a constant K such that $d(x, y) \leqslant K(d(x, z) + d(z, y))$ for every $x, y, z \in X$, and $d(x, y) = 0$ if and only if $x = y$. Let μ be a positive measure defined on a σ-algebra of subsets of X which contains the d-balls and satisfies the following: there exists a constant C such that $0 < \mu(B(x, 2r)) \leqslant C\mu(B(x, r)) < \infty$ holds for every $x \in X$ and $r > 0$. We shall say that (X, d, μ) is a space of homogeneous type if X is a set endowed with a quasi-distance and a measure satisfying these conditions. If $B = B(x, r)$ is a d-ball in X, we write $\tilde{B}$ for $B(x, 5K^2r)$ and $\hat{B}$ for $B(x, 15K^5r)$. Let A be such that $\mu(\hat{B}) \leqslant A\mu(B)$ for every B. If f is a positive measurable function defined on X and E a measurable set, $f(E)$ means $\int_E f\,d\mu$, $m_E f = \mu(E)^{-1} \cdot f(E)$ and $Mf(x) = \sup m_B|f|$, where the supremum is taken over all balls B containing x. By modifying slightly the proof of Theorem (1.2) in Chapter III of [4], we get the following covering lemma.

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LEMMA 1. *Let E be a bounded subset of X and assume that for each $x \in E$ there exist $y(x) \in X$ and $r(x) > 0$ such that $x \in B(y(x), r(x))$. Then, there exists a sequence of disjoint balls $\{B(y(x_i), r(x_i))\}$ such that $E \subset \bigcup_{i=1}^{\infty} B(y(x_i), 5K^2 r(x_i))$.*

LEMMA 2. *Suppose $\mu(X) = \infty$. For any nonnegative integrable function f with bounded support, $b \geq 2A^3 > 1$ and any $k \in \mathbf{Z}$ such that $\Omega_k = \{y \in X: b^{k+1} \geq Mf(y) > b^k\} \neq \varnothing$, there exists a sequence of balls $\{B_i^k\}_{i \in \mathbf{N}}$ satisfying*

(2.1) $\Omega_k \subset \bigcup_{i=1}^{\infty} \tilde{B}_i^k$.

(2.2) $B_i^k \cap B_j^k = \varnothing$ *if* $i \neq j$,

(2.3) *For every B_i^k, there exists $x_i^k \in B_i^k$ such that if r_i^k is the radius of B_i^k, $r \geq 5K^2 r_i^k$, and $x_i^k \in B(y, r) = B$, then $b^{k+1} \geq Mf(x_i^k) \geq m_{B_i^k} f > b^k \geq m_B f$.*

(2.4) *If $x \notin \bigcup_{j=k}^{\infty} \bigcup_{i=1}^{\infty} \tilde{B}_i^j$ and $Mf(x) < \infty$, then $Mf(x) \leq b^k$.*

(2.5) *Let $I_j^k = \{(l, n) \in \mathbf{Z} \times \mathbf{N}: l \geq k+2, \tilde{B}_n^l \cap \tilde{B}_j^k \neq \varnothing\}$ and let $A_j^k = \bigcup_{(l,n) \in I_j^k} \tilde{B}_n^l$, then $2\mu(A_j^k) \leq \mu(B_j^k)$.*

(2.6) *Let $E_j^k = \tilde{B}_j^k - A_j^k$, then $2\mu(E_j^k) \geq \mu(\tilde{B}_j^k)$ and $\mu(X - \bigcup_{k,j} E_j^k) = 0$. If $x \in E_j^k$ and $Mf(x) < \infty$, then $Mf(x) \leq b^{k+2}$.*

(2.7) *If $F_j^k = B_j^k - A_j^k$, then $\mu(F_j^k) \geq \mu(\tilde{B}_j^k)/2A$ and $\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq 3$, where χ_E denotes the characteristic function of the set E.*

PROOF. If $x \in \Omega_k$, the integrability of f implies that the set $R_k(x) = \{r > 0: m_B f > b^k,\ x \in B = B(y, r)\}$ is bounded. We can choose $r(x) \in R_k(x)$ in such a way that if $r \geq 5K^2 r(x)$, then $r \notin R_k(x)$. Thus, there is a point $y(x) \in X$ such that

$$b^{k+1} \geq Mf(x) \geq m_{B(y(x), r(x))} f > b^k \geq m_{B(y,r)} f,$$

whenever $r \geq 5K^2 r(x)$ and $x \in B(y, r)$. The boundedness of the support of f implies that of Ω_k, therefore Lemma 1 can be applied to obtain a sequence $\{B_i^k\}$ satisfying (2.1)–(2.4). In order to get (2.5), let us first show that if $l \geq k+2$, $n \in \mathbf{N}$ and $\tilde{B}_n^l \cap \tilde{B}_j^k \neq \varnothing$, then

$$\tilde{B}_n^l \subset \hat{B}_j^k, \tag{2.8}$$

even more $r_n^l \leq r_j^k$. Indeed, if we suppose that $r_n^l > r_j^k$, then $\tilde{B}_j^k \subset \hat{B}_n^l$. The last inequality in (2.3) applied to $B(y, r) = \hat{B}_n^l$ gives

$$b^k \geq m_{\hat{B}_n^l} f \geq \mu(B_n^l) \mu(\hat{B}_n^l)^{-1} m_{B_n^l} f \geq A^{-1} m_{B_n^l} f,$$

by the third inequality in (2.3) applied to the pair (l, n), we have $A^{-1} m_{B_n^l} f > A^{-1} b^l \geq A^{-1} b^{k+2}$, which is a contradiction. Now (2.3), (2.2) and (2.8) yield (2.5) in the following way:

$$\mu(A_j^k) \leq \sum_{I_j^k} \mu(\tilde{B}_n^l) \leq A \sum_{I_j^k} b^{-l} \int_{B_n^l} f \, d\mu \leq A \left(\sum_{l=k+2}^{\infty} b^{-l} \right) \int_{\hat{B}_j^k} f \, d\mu$$

$$\leq A^2 b^{-k-1} (b-1)^{-1} \mu(B_j^k) m_{\hat{B}_j^k} f \leq \mu(B_j^k)/2.$$

In order to prove (2.6), let x be a point such that $Mf(x) < \infty$; then $x \in \Omega_k$ for some $k \in \mathbf{Z}$. By (2.1), $x \in \tilde{B}_j^k$ for some $j \in \mathbf{N}$. Assume that $x \in A_j^k$, then there exists $(l, n) \in I_j^k$ such that $x \in \tilde{B}_n^l$, and from (2.3) we obtain

$$Mf(x) \geq m_{\tilde{B}_n^l} f \geq A^{-1} m_{B_n^l} f > A^{-1} b^{k+2} > b^{k+1},$$

which is a contradiction. Thus, the sequence $\{E_j^k\}$ is a covering of $\{x\colon Mf(x) < \infty\}$. On the other hand, on account of the weak type (1, 1) boundedness of the Hardy-Littlewood maximal function operator, the set $\{x\colon Mf(x) = \infty\}$ is of measure zero and therefore (2.6) is proved. From (2.2) we see that

$$\sum_{j=1}^{\infty} \chi_{F_j^k}(x) = \chi_{\cup_{j=1}^{\infty} F_j^k}(x) \leqslant \chi_{\cup_{j=1}^{\infty} E_j^k}(x),$$

for any $k \in \mathbf{Z}$. By definition of E_j^k it follows readily that no point of X belongs to more than three of the sets $\bigcup_{j=1}^{\infty} E_j^k$. Then

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leqslant \sum_{k=-\infty}^{\infty} \chi_{\cup_{j=1}^{\infty} E_j^k}(x) \leqslant 3,$$

which is (2.7). This finishes the proof of the lemma.

With this result the argument given in **[1]** can be adapted to our purposes and we shall check the details. Let w be a weight function satisfying the A_p condition

$$\sup_B m_B w\left[m_B w^{1/(1-p)}\right]^{p-1} < \infty.$$

As in the euclidean case, the weight $\sigma = w^{1/(1-p)}$ satisfies A_q where $q + p = qp$; then both w and σ satisfy the A_∞ condition, that is

$$\mu(E)\mu(B)^{-1} \leqslant Cw(E)^{\delta} w(B)^{-\delta},$$

for all $E \subset B$, with C and δ positive and independent of E and B. Suppose $\mu(X) = \infty$ and f as in Lemma 2. Then (2.6) and (2.3) yield

$$\begin{aligned}\int_X (Mf)^p w\, d\mu &\leqslant b^{2p} \sum_{k,j} \left(m_{B_j^k} f\right)^p w\left(E_j^k\right)\\ &\leqslant b^{2p} \sum_{k,j} \left[\frac{1}{\sigma\left(B_j^k\right)} \int_{B_j^k} (f\sigma^{-1})\sigma\, d\mu\right]^p\\ &\quad \times \sigma\left(E_j^k\right) \frac{\sigma\left(\tilde{B}_j^k\right)}{\sigma\left(E_j^k\right)} \left\{\left[\frac{\sigma\left(B_j^k\right)}{\mu\left(B_j^k\right)}\right]^{p-1} \cdot \frac{w\left(\tilde{B}_j^k\right)}{\mu\left(B_j^k\right)}\right\}.\end{aligned}$$

Applying the A_∞ condition on w and σ and the A_p condition on w, the $L^p(w\, d\mu)$ norm of Mf is bounded by

$$C\left\{\sum_{k,j} \left[\frac{1}{\sigma\left(B_j^k\right)} \int_{B_j^k} (f\sigma^{-1})\sigma\, d\mu\right]^p \sigma\left(E_j^k\right)\right\}^{1/p}.$$

Applying (2.7) and the A_∞ condition on σ, this is bounded by

$$C\left\{\sum_{k,j} \left[\frac{1}{\sigma\left(B_j^k\right)} \int_{B_j^k} (f\sigma^{-1})\sigma\, d\mu\right]^p \int_{F_j^k} \sigma\, d\mu\right\}^{1/p} \leqslant C\left\{\int_X \left[M_\sigma(f\sigma^{-1})\right]^p \sigma\, d\mu\right\}^{1/p},$$

where M_σ is the Hardy-Littlewood maximal function operator on the space of homogeneous type $(X, d, \sigma\, d\mu)$. Then, we have

$$\|Mf\|_{L^p(w d\mu)} \leqslant C\left\{\int_X f^p \sigma^{-p+1}\, d\mu\right\}^{1/p} = C\|f\|_{L^p(w d\mu)}.$$

This completes the proof for the case $\mu(X) = \infty$. If $\mu(X) < \infty$, let Y be $X \times \mathbf{R}$, δ: $Y \times Y \to \mathbf{R}^+ \cup \{0\}$ defined by $\delta((x_1, t_1), (x_2, t_2)) = \max\{d(x_1, x_2), |t_1 - t_2|\}$ and $\nu = \mu \times \lambda$, where λ is the Lebesgue measure on $\mathbf{R}$; then (Y, d, ν) is a space of homogeneous type with $\nu(Y) = \infty$. Given a weight w satisfying A_p on X, then $W(x, t) = w(x)$ satisfies A_p on Y. If f is a measurable function on X, define $F(x, t) = f(x)\chi_{(-2R, 2R)}(t)$ on Y, where R is such that $B(x, R) = X$ for every $x \in X$. With these definitions it is clear that $Mf(x) \leqslant M_Y F(x, t)$ for all $t \in (-R, R)$, where M_Y is the Hardy-Littlewood maximal function operator on Y. By these remarks, the result just proved applied to M_Y, F and W implies the desired inequality.

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PROGRAMA ESPECIAL DE MATEMATICA APLICADA, CONICET, C.C. N°. 91, 3000 SANTA FE, ARGENTINA