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ON BOUNDEDNESS OF COMPOSITION OPERATORS ON $H^2(B_2)$

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ABSTRACT. Composition operators on the Hardy space H^2 of the ball in $\mathbf{C}^2$ are studied. Some sufficient conditions are given for a composition operator to be bounded. A class of inner mappings is given which induces isometric composition operators. Another class of inner mappings is shown to induce unbounded composition operators.

1. Introduction and definitions. Let B_n be the unit ball in $\mathbf{C}^n$ and let $H^p = H^p(B_n)$ be the Hardy space on B_n. For $p \geqslant 1$, H^p is a Banach space. If $\phi\colon B_n \to B_n$ is holomorphic, then ϕ induces the linear operator C_ϕ from H^p into the holomorphic functions on B_n given by $C_\phi f = f \circ \phi$. C_ϕ is the composition operator induced by ϕ.

Definitive results on composition operators are known for the case $n = 1$ (see **[3]** and the references given therein). In particular, for $n = 1$, C_ϕ is always a bounded operator on H^p. One can check the integral condition on $f \circ \phi$ or observe that the class of positive harmonic functions is invariant under holomorphic change of variables. In addition, subordination techniques can be used to compare the integral means of f and $f \circ \phi$.

The proofs that every C_ϕ is bounded for $n = 1$ fail if $n \geqslant 2$. In fact, J. H. Shapiro has given examples of holomorphic mappings ϕ of B_n into B_n ($n \geqslant 2$) for which C_ϕ is not bounded on H^2. Note that although we still have the existence of certain positive majorants (harmonic, M-harmonic), these classes of majorants are not invariant under holomorphic change of variables.

The results of this paper are of two types. We show that a certain class of inner mappings induces unbounded composition operators on H^2. We also give some sufficient conditions that C_ϕ be bounded. We give examples in §4 which answer in the affirmative the following question of Rudin **[5]**: "Is there an inner mapping ϕ of B_n into B_n, not an automorphism of B_n, which is measure-preserving?" If ϕ is measure-preserving, then C_ϕ is an isometry from H^p into H^p. We remark that for $1 \leqslant p \leqslant \infty$, $p \neq 2$, the onto isometries of H^p are known (cf. **[4]**, p. 152]).

Finally, we note that B. MacCluer **[1]** has studied C_ϕ when ϕ is a holomorphic automorphism of B_n. She has also studied the spectra of compact composition operators and has given some examples of unbounded composition operators in **[2]**.

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We restrict our attention to $p = 2$ and, for notational convenience, to $n = 2$. Let $B = \{z = (z_1, z_2): |z_1|^2 + |z_2|^2 < 1\}$ and let S be the boundary of B. Let $d\sigma$ denote normalized surface measure on S and let dv denote volume measure on B. A holomorphic function f on B is in H^2 if

$$\sup_{0<r<1} \int_S |f(r\zeta)|^2 \, d\sigma(\zeta) \equiv \|f\|^2 < \infty.$$

It is known [**4**, p. 84] that f is in H^2 if and only if $|f|^2$ has a harmonic majorant if and only if $|f|^2$ has an M-harmonic majorant. (Recall that u: $B \to \mathbf{C}$ is M-harmonic if $\tilde{\Delta} u = 0$, where $\tilde{\Delta}$ is the invariant Laplacian [**4**, p. 47].) In the latter case, there is a least M-harmonic majorant of the form

$$u(z) = \int_S P(z, \zeta) h(\zeta) \, d\sigma(\zeta), \qquad z \in B, h \in L^2(S),$$

and

$$P(z, \zeta) = \left((1 - |z|^2)/|1 - \langle z, \zeta \rangle|^2\right)^2.$$

All mappings are assumed to be holomorphic. A mapping ϕ: $B \to B$ is inner if for almost every $\zeta \in S$, the radial limit $\lim_{r \to 1} \phi(r\zeta) = \phi(\zeta)$ is in S.

Finally, we recall [**3**] that if ϕ is a mapping of B_1 to B_1, then

$$\|C_\phi\|^2 \leqslant (1 + |\phi(0)|)/(1 - |\phi(0)|). \tag{1}$$

2. Unbounded composition operators. The first example known to us of a mapping ϕ for which C_ϕ is unbounded was given by J. H. Shapiro (private communication). Let $\phi(z_1, z_2) = (2z_1 z_2, 0)$. Then ϕ maps B onto a slice, and ϕ has vanishing Jacobian. Further, for $\zeta \in S$, we have $|\phi(\zeta)| = 1$ precisely when ζ is on the torus $\{2^{-1/2}(e^{i\theta_1}, e^{i\theta_2}): \theta_1, \theta_2 \in [0, 2\pi)\}$. We sketch a proof that C_ϕ is unbounded. The set $\{f_{j,k}(z_1, z_2) = z_1^j z_2^k: j, k \geqslant 0\}$ is an orthogonal basis for H^2, and

$$\|f_{j,k}\|^2 = j!k!/(1 + j + k)!.$$

Now $C_\phi f_{j,0} = 2^j f_{j,j}$, and an application of Stirling's formula will show that

$$\lim_{j \to \infty} \frac{\|C_\phi f_{j,0}\|}{\|f_{j,0}\|} = \infty.$$

(See [**2**] for a more general version of this example.)

We present another simple example of an unbounded composition operator. Let ψ be an inner function and let $\phi(z_1, z_2) = (\psi(z_1, z_2), 0)$. Then C_ϕ is unbounded. (This example also appears in [**2**] with a different proof of unboundedness.)

Since automorphisms of B induce bounded composition operators, we may assume $\phi(0) = 0$. Choose $f \in H^2$ for which $\int_0^{2\pi} |f(e^{i\theta}, 0)|^2 \, d\theta = \infty$. For $\zeta \in S$, let $\phi_\zeta(e^{i\theta}) = \phi(e^{i\theta}\zeta)$. Then for almost every ζ, ϕ_ζ is inner and $\phi_\zeta(0) = 0$. Thus (see [**3**])

ϕ_ζ induces an isometry on $H^2(B_1)$. Hence (see [**4**, Proposition 1.4.7(1)]),

$$\int_S |f\circ\phi|^2\,d\sigma = \int_S d\sigma(\zeta)\,\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(\phi_\zeta(e^{i\theta})\right)\right|^2 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\theta},0)\right|^2 d\theta = \infty.$$

We generalize this example as follows.

THEOREM 1. *Suppose $(A, B) \in S$ and ϕ_1 and ϕ_2 are inner functions on B. If $\phi(z_1, z_2) = (A\phi_1(z_1, z_2), B\phi_2(z_1, z_2))$, then C_ϕ is unbounded.*

PROOF. The theorem is true for $A = 0$ or $|A| = 1$, so let $r = |A|^2$ and suppose $0 < r < 1$. For every positive integer N, there is an integer K_N so that $(K_N - 1)/N \leqslant r < K_N/N$. Thus $(K_N - 1)/K_N \leqslant Nr/K_N < 1$, so

$$(1 - 1/K_N)^{K_N} < (Nr/K_N)^{K_N} < 1. \tag{2}$$

Choose N_0 so that if $N \geqslant N_0$, then $K_N \geqslant 2$. For $N \geqslant N_0$, we have $\frac{1}{4} < (1 - 1/K_N)^{K_N}$. Thus by (2), $r^{K_N} > \frac{1}{4}(K_N/N)^{K_N}$. Also note that $1 - K_N/N < 1 - r$. Then

$$\left\|f_{K_N,N-K_N}\circ\phi\right\|^2 = r^{K_N}(1-r)^{N-K_N} \geqslant \frac{1}{4}\left(\frac{K_N}{N}\right)^{K_N}\left(\frac{N-K_N}{N}\right)^{N-K_N}. \tag{3}$$

We use the following form of Stirling's formula:

$$n! = (n^n/e^n)\sqrt{n}\,e^{c_n}, \quad \text{with } e \geqslant e^{c_n} \geqslant e/\sqrt{2}\,.$$

Now by (3) and Stirling's formula,

$$\frac{\left\|f_{K_N,N-K_N}\circ\phi\right\|^2}{\left\|f_{K_N,N-K_N}\right\|^2} \geqslant C_N\left(\frac{1}{4}\right)\left(\frac{K_N}{N}\right)^{K_N}\left(\frac{N-K_N}{N}\right)^{N-K_N}$$
$$\times\frac{(N+1)^{N+1}\sqrt{N+1}\,e^{K_N}e^{N-K_N}}{e^{N+1}(K_N)^{K_N}\sqrt{K_N}\,(N-K_N)^{N-K_N}\sqrt{N-K_N}}$$
$$\geqslant \frac{C_N}{4}\left(\frac{N+1}{N}\right)^N\frac{(N+1)^{3/2}}{\sqrt{K_N(N-K_N)}} > \frac{C_N}{4}\sqrt{N}\,.$$

Hence C_ϕ is unbounded.

We remark that we can choose the inner mapping ϕ of Theorem 1 so that its Jacobian is not identically zero. However, the range of ϕ is contained in a polydisc, so that ϕ is not onto B.

3. Bounded composition operators. The linear mapping $\phi(z_1, z_2) = (z_1, 0)$ has the same range as Shapiro's example, has vanishing Jacobian, and induces a bounded composition operator. This can be deduced by [**4**, p. 127] or from the following proposition. (See [**5**] for the polydisc version.)

PROPOSITION 1. *If ϕ is a mapping from B into B and if $\phi(z_1, z_2) = (\phi_1(z_1), \phi_2(z_2))$, then C_ϕ is bounded.*

PROOF. As before, we may assume $\phi(0) = 0$. Fix $r < 1$ and let $f \in H^2$. Then (see [**4**, p. 15])

$$\int_S |f \circ \phi(r\zeta)|^2 \, d\sigma(\zeta) = \int_S \left| \frac{1}{2\pi} \int_0^{2\pi} f\left(\phi_1\left(re^{i\theta}\zeta_1\right), \phi_2(r\zeta_2)\right) d\theta \right|^2 d\sigma(\zeta).$$

Fix ζ_2 and apply (1). Then

$$\int_S |f \circ \phi(r\zeta)|^2 \, d\sigma(\zeta) \leqslant \int_S \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\zeta_1, \phi_2(r\zeta_2)\right)\right|^2 d\theta \, d\sigma(\zeta).$$

Repeating this argument in the second coordinate gives $\|f \circ \phi\|^2 \leqslant \|f\|^2$.

A mapping on B of the form $\phi(z_1, z_2) = (z_1^j/a, z_2^k/b)$ has a "generalized ellipsoid" as its range. If ϕ maps into B, then Proposition 1 shows C_ϕ is bounded.

The following theorem gives another sufficient condition for boundedness of C_ϕ. We use the notation $J\phi = (D_i\phi_j)_{i,j=1}^2$ for the complex Jacobian matrix of the mapping ϕ, and $|J\phi|$ is the absolute value of the determinant of $J\phi$. Also Δ denotes the Laplacian.

THEOREM 2. *Let ϕ be a one-to-one mapping of B into B. Suppose there are positive constants c and M so $|J\phi(z)| \geqslant c$ on B and $|D_i\phi_j(z)| \leqslant M$ on B, for $i, j = 1, 2$. Then C_ϕ is bounded.*

PROOF. As before, we assume $\phi(0) = 0$. Assume f is holomorphic on B and $0 < r < 1$. Let $g_r(z) = |z|^{-2} - r^{-2}$ and let $B(r) = \{z \in \mathbf{C}^2 \colon |z| < r\}$. Apply Green's Theorem to conclude that

$$4\pi^2 \int_S \left(g_r(r\zeta) \frac{\partial |f(r\zeta)|^2}{\partial r} - |f(r\zeta)|^2 \frac{\partial g_r(r\zeta)}{\partial r} r^3 \right) d\sigma(\zeta) = \int_{B(r)} \left(g_r \Delta |f|^2 - |f|^2 \Delta g_r \right) dv$$

Then

$$8\pi^2 \int_S |f(r\zeta)|^2 \, d\sigma(\zeta) = \int_{B(r)} g_r \Delta |f|^2 \, dv + 8\pi^2 |f(0)|^2.$$

By integration by parts we obtain

$$\int_S |f(r\zeta)|^2 \, d\sigma(\zeta) = \frac{1}{8\pi^2} \int_0^r \rho^{-3} \left[\int_{B(\rho)} \Delta |f|^2 \, dv \right] d\rho + |f(0)|^2. \tag{3}$$

Now $\phi(B(r)) \subset B(r)$ by the Schwarz Lemma, and $\Delta |f|^2 \geqslant 0$, so

$$\int_{B(r)} \Delta |f|^2 \, dv \geqslant \int_{\phi(B(r))} \Delta |f|^2 \, dv = \int_{B(r)} \left[\Delta |f|^2 \circ \phi \right] |J\phi|^2 \, dv$$

$$= \sum_{j=1}^2 \int_{B(r)} |D_j f|^2(\phi) |J\phi|^2 \, dv.$$

Also

$$\int_{B(r)} \Delta|f \circ \phi|^2 \, dv = \int_{B(r)} \sum_{j=1}^{2} \left| \sum_{i=1}^{2} D_i f(\phi) D_j \phi_i \right|^2 dv.$$

Now we use our hypothesis to conclude that

$$\frac{C^2}{4M^2} \int_{B(r)} \Delta|f \circ \phi|^2 \, dv \leq \int_{B(r)} \Delta|f|^2 \, dv.$$

Hence from (3), there is a constant A so that $\|f \circ \phi\|^2 \leq A\|f\|^2$.

Note that equation (3) in the proof of Theorem 2 is Jensen's formula for subharmonic functions applied to $|f|^2$.

EXAMPLE. Let g be a holomorphic function from B into the unit disc and define $\phi(z_1, z_2) = g(z_1, z_2)(z_1, z_2)$. ϕ maps B into B. If $\zeta \in S$, then ϕ maps the slice $\{\lambda\zeta: |\lambda| \leq 1\}$ into itself. If $f \in H^2$ and $f_\zeta(\lambda) = f(\lambda\zeta)$, then

$$\|f\|^2 = \int_S |f(\zeta)|^2 \, d\sigma(\zeta) = \int_S \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}\zeta)|^2 \, d\theta \, d\sigma(\zeta) = \int_S \|f_\zeta\|^2 \, d\sigma(\zeta).$$

For fixed $\zeta \in S$, we have $f(\phi(\lambda\zeta)) = f_\zeta(\lambda g_\zeta(\lambda))$. So by (1),

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f_\zeta\left(e^{i\theta} g_\zeta(e^{i\theta})\right) \right|^2 d\theta \leq \|f_\zeta\|^2.$$

Thus $\|f \circ \phi\|^2 \leq \|f\|^2$, and $\|C_\phi\| = 1$.

If the function g of the Example is an inner function, then ϕ is an inner mapping which is not an automorphism but such that C_ϕ is bounded. In fact, we will see from Theorem 3 that C_ϕ is an isometry.

4. Measure-preserving inner mappings. An inner mapping ϕ is called measure-preserving if for all Borel subsets E of S, we have $\sigma(E) = \sigma(\phi^{-1}(E))$.

PROPOSITION 2. *ϕ is measure-preserving if and only if*

$$\int_S f_{j,k}(z) \bar{f}_{m,n}(z) \, d\sigma(z) = \int_S f_{j,k}(\phi(z)) \bar{f}_{m,n}(\phi(z)) \, d\sigma(z), \tag{4}$$

for all j, k, m, $n \geq 0$.

The proof is straightforward. Use change of variables, the Riesz Representation Theorem for $C(S)$, and the Stone-Weierstrass Theorem.

Since the $f_{j,k}$'s form an orthogonal basis for H^2, we have

COROLLARY 1. *ϕ is measure-preserving if and only if C_ϕ is an isometry.*

LEMMA 1. *If $\psi \in H^\infty$ and either $j > m$ or $k > n$, then $\langle f_{j,k}\psi, f_{m,n} \rangle = 0$. ($\langle \cdot, \cdot \rangle$ denotes the inner product in H^2.)*

PROOF. Let ν denote area measure in **C**. Suppose $j > m$. Then

$$\langle f_{j,k}\psi, f_{m,n}\rangle = \int_{|z_2|<1} z_2^k \bar{z}_2^n \left(1 - |z_2|^2\right)^{(j+m)/2} \times \left[\frac{1}{2\pi}\int_0^{2\pi} \psi\left(\sqrt{1-|z_2|^2}\, e^{i\theta}, z_2\right) e^{i(j-m)\theta}\, d\theta\right] d\nu(z_2) = 0,$$

since the inside integral vanishes. The case $k > n$ is handled similarly.

THEOREM 3. *Suppose ψ is an inner function on B and M and N are nonnegative integers. If $\phi(z_1, z_2) = (z_1\psi^M(z_1, z_2), z_2\psi^N(z_1, z_2))$, then ϕ is a measure-preserving inner mapping.*

PROOF. First note that $f_{j,k} \circ \phi = f_{j,k}\psi^{Mj+Nk}$. If $(j, k) = (m, n)$, then (4) clearly holds. If $(j, k) \neq (m, n)$, then

$$\langle f_{j,k} \circ \phi, f_{m,n} \circ \phi\rangle = \langle f_{j,k}\psi^{M(j-m)+N(k-n)}, f_{m,n}\rangle.$$

If the exponent on ψ is > 0, then Lemma 2 shows that this inner product is $0 = \langle f_{j,k}, f_{m,n}\rangle$. Thus (4) holds. If the exponent is < 0, write the inner product as $\langle f_{j,k}, f_{m,n}\psi^{M(m-j)+N(n-k)}\rangle$ and apply Lemma 1.

We remark that the class of measure-preserving inner mappings is a semigroup under composition. Thus we can use Theorem 3 to generate other examples of measure-preserving maps (and isometric composition operators).

Finally, we note that all of the questions considered in this paper have analogues for the polydisc. One can show, for instance, that not every holomorphic self-map of the polydisc induces a bounded composition operator. In fact (see [**6**]) if $\phi(z_1, z_2) = (z_1, z_1)$, then C_ϕ is not bounded on H^2 of the bidisc.

ADDED IN PROOF. B. Tomaszewski has shown in a preprint entitled *Interpolation and inner maps that preserve measure*, that for $n \geqslant m \geqslant 1$ there exist inner mappings Φ; $B^n \to B^m$ which are measure-preserving.

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