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THE CARATHÉODORY DISTANCE DOES NOT DEFINE THE TOPLOGY

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ABSTRACT. We construct an analytic space X such that the Carathéodory pseudodistance c_X is a true distance on X; however, c_X does not define the analytic space topology of X.

1. Introduction. In [**2**], S. Kobayashi gives a list of problems about intrinsic distances on an analytic space. Some of these problems have been solved by T. Barth [**1**] and the author [**4**]. In a similar vein, T. Barth [**1**] and N. Sibony [**3**] asked the following question: Let X be a finite-dimensional analytic space for which the Carathéodory distance c_X is a true distance. Does c_X define the analytic space topology of X? In [**3**], N. Sibony proved that the answer is affirmative provided X is finitely compact with respect to c_X. In this paper we shall prove that, in general, the answer is negative.

2. The example. Take real numbers α, β, γ, δ such that $0 < \beta < \alpha < 1$, $0 < \gamma < \alpha < 1$, and $0 < \delta < 1$. Let D be the union of these three open sets in $\mathbf{C}^2$ (coordinates z_1, z_2):

$$D_1 = \left\{(z_1, z_2) \in \mathbf{C}^2 \big| |z_1| < 1, |z_2| < \beta\right\};$$
$$D_2 = \left\{(z_1, z_2) \in \mathbf{C}^2 \big| 1 > |z_1| > \delta, |z_2| < 1\right\};$$
$$D_3 = \left\{(z_1, z_2) \in \mathbf{C}^2 \big| |z_1| < 1, \alpha < |z_2| < 1\right\}.$$

Similarly, let E be the union of these three open sets in $\mathbf{C}^2$ (coordinates z_2, z_3):

$$E_1 = \left\{(z_2, z_3) \in \mathbf{C}^2 \big| 0 < |z_3| < 1, |z_2| < \gamma\right\};$$
$$E_2 = \left\{(z_2, z_3) \in \mathbf{C}^2 \big| 1 > |z_3| > \delta, |z_2| < 1\right\};$$
$$E_3 = \left\{(z_2, z_3) \in \mathbf{C}^2 \big| |z_3| < 1, \alpha < |z_2| < 1\right\}.$$

Let X be the analytic space obtained by patching together D and E as two planes cutting transversely along the z_2-axis for $\alpha < |z_2| < 1$. It is easy to see that the Carathéodory pseudodistance c_X is a true distance on X. However, by using the extension theorem for holomorphic functions, we can prove the following. Let x_n be the image in X of the point $(0, 1/n)$ in E. Let 0 be the image in X of the origin $(0, 0)$

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in D. Then $c_X(x_n, 0) \to 0$, but x_n does not converge to 0 in the analytic space topology. Thus we have proved the following theorem.

THEOREM. *On X, the Carathéodory pseudodistance c_X is a distance, but c_X does not define the analytic space topology.*

REMARK 1. Unfortunately, I do not know how to construct an example in which X is a complex manifold.

REMARK 2. By [**1**] this shows that the Carathéodory distance c_X is not inner, answering a question in [**2**]. Of course, this question is already answered in [**1** and **4**]. In fact, in [**4**] I constructed a bounded domain, finitely compact with respect to its Carathéodory distance, such that this distance is not inner.

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