

EQUIVARIANT TRIVIALITY THEOREMS FOR HILBERT C^* -MODULES

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ABSTRACT. The purpose of this paper is to give an exposition of the various triviviality theorems, the equivariant version of a result due to L. Brown, and a simplification of the proof of Kasparov's triviviality theorems.

0. Introduction and notation. In [5] several triviviality theorems are given for continuous fields of Hilbert spaces $(\mathcal{H}(z), \Gamma)$ over a paracompact space B . When B is locally compact and \mathcal{E} is the subspace of Γ of functions vanishing at infinity, then \mathcal{E} is a Hilbert $C_0(B)$ -module.

Recently some of these triviviality theorems [5, Théorème 4 and Corollaire 3] have been generalized to the case of Hilbert C^* -modules for noncommutative algebras [2, 6, 7, 9]. Our purpose is to give an exposition of the various triviviality theorems, the equivariant version of the triviviality theorem of [2], and a simplification of the proof of Kasparov's triviviality theorems [7, 9].

Although Hilbert C^* -modules had been considered earlier than [7] (see e.g. [10]), we will adopt the notation of Kasparov [7, §2, Definitions 1–4]. If \mathcal{E} is a Hilbert A -module then \mathcal{E}^∞ denotes the direct sum of \mathcal{E} with itself countably many times; an isomorphism of Hilbert A -modules is denoted by \simeq . \mathcal{K}_A denotes A^∞ where A is considered a module over itself [7, §2, Example 1].

The two triviviality theorems then are

THEOREM 1.4 [5, 6, 7, 9]. *Let \mathcal{E} be a countably generated Hilbert A -module; then $\mathcal{E} \oplus \mathcal{K}_A \simeq \mathcal{K}_A$.*

THEOREM 1.9 [2, 5, 6]. *Let \mathcal{E} be a full countably generated Hilbert A -module. If A has a strictly positive element, then $\mathcal{E}^\infty \simeq \mathcal{K}_A$.*

1. Triviviality theorems without group actions. In this section we consider the triviviality theorems mentioned in §0 but without any group actions. A crucial notion in this section is that of a strictly positive element.

DEFINITION 1.1 [1]. If e is a positive element of a C^* -algebra A and $\phi(e) \neq 0$ for all states ϕ on A , then e is strictly positive.

The following lemma, observed in [2], can be deduced from [1], but since it has a straightforward proof, we give it here.

LEMMA 1.2. *If e is a positive element of A then e is strictly positive if and only if eA is dense in A .*

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It is clear that $\text{ran}(K)$ is dense so K is strictly positive. Thus T^*T is strictly positive. So $\text{ran}(T^*T)$ is dense and thus $\text{ran}(|T|)$ is also dense. Finally, define $V: \mathcal{K}_A \rightarrow \mathcal{E} \oplus \mathcal{K}_A$ by $V(|T|\xi) = T\xi$. As $\|V(|T|\xi)\| = \||T|\xi\|$, V has a continuous extension to \mathcal{K}_A , where it becomes a unitary from \mathcal{K}_A to $\mathcal{E} \oplus \mathcal{K}_A$. Q.E.D.

COROLLARY 1.5. *If \mathcal{E} is a Hilbert A -module then \mathcal{E} is countably generated if and only if $\mathcal{K}(\mathcal{E})$ has a strictly positive element.*

PROOF. As in the proof of Theorem 1.4 we may suppose A is unital. By Theorem 1.4 there is a projection P in $\mathcal{L}(\mathcal{K}_A)$ with $\mathcal{E} \cong P(\mathcal{K}_A)$. Let $\{\xi_n\}$ be the standard orthonormal basis for \mathcal{K}_A . Then $K = \sum 1/n\theta_{\xi_n, \xi_n}$ is a strictly positive element of $\mathcal{K}(\mathcal{K}_A)$ by Lemma 1.3. Now $\mathcal{K}(\mathcal{E}) \cong P\mathcal{K}(\mathcal{K}_A)P$, so $\mathcal{K}(\mathcal{E})$ has a strictly positive element [3, Proposition 2.3], PKP.

Conversely, if $\mathcal{K}(\mathcal{E})$ has strictly positive element $K = \sum_{i=1}^\infty \theta_{\xi_i, \eta_i}$ with $\xi_i, \eta_i \in \mathcal{E}$, then as $K\mathcal{E}$ is dense, $\{\xi_i\}_{i=1}^\infty$ is a set of generators.

DEFINITION 1.6. *If \mathcal{E} is a Hilbert A -module then $\langle \mathcal{E}, \mathcal{E} \rangle = \{\sum \langle \xi_i, \eta_i \rangle : \xi_i, \eta_i \in \mathcal{E}\}$ is called the support of \mathcal{E} . \mathcal{E} is full if $\langle \mathcal{E}, \mathcal{E} \rangle = A$.*

LEMMA 1.7. *If \mathcal{E} is a full Hilbert A -module and A has a strictly positive element then there is a sequence $\{\xi_i\}$ in \mathcal{E} such that $\sum \langle \xi_i, \xi_i \rangle = 1$ strictly in $M(A)$.*

PROOF. This is precisely the statement of Lemma 2.3 of [2] when $\mathcal{E} = pA$ and $\langle \xi, \eta \rangle = \xi^*\eta$ for p a projection in $M(A)$ and $\xi, \eta \in \mathcal{E}$. The proof goes over to the more general case with obvious modifications. Q.E.D.

COROLLARY 1.8. *If \mathcal{E} is a full Hilbert A -module and A has strictly positive element then $\mathcal{E}^\infty \simeq A \oplus \mathcal{F}$ for some Hilbert A -module \mathcal{F} .*

PROOF. Let $\{\xi_i\}$ be as in Lemma 1.7. Define $T: A \rightarrow \mathcal{E}^\infty$ by $T(a) = (\xi_i a)$. As $\langle (\xi_i a), (\xi_i a) \rangle = a^*a$, we see that $(\xi_i a) \in \mathcal{E}^\infty$. Define $T^*: \mathcal{E}^\infty \rightarrow A$ by $T^*(\eta_i) = \sum \langle \xi_i, \eta_i \rangle$. By applying the Cauchy-Schwarz inequality we see that $\sum \langle \xi_i, \eta_i \rangle$ converges in norm to an element of A . As $T^*T = \text{id}_A$ we have that $T \oplus \text{id}: A \oplus (1 - TT^*)\mathcal{E}^\infty \rightarrow \mathcal{E}^\infty$ is an isomorphism. Q.E.D.

THEOREM 1.9. *If \mathcal{E} is a countably generated full Hilbert A -module and A has a strictly positive element, then $\mathcal{E}^\infty \simeq \mathcal{K}_A$.*

PROOF. $\mathcal{E}^\infty \simeq (A \oplus \mathcal{F})^\infty = \mathcal{K}_A \oplus \mathcal{F}^\infty \simeq \mathcal{K}_A$, where the last isomorphism follows from the stabilization theorem because \mathcal{F} , being a complemented submodule of \mathcal{E}^∞ , is countably generated. Q.E.D.

REMARK 1.10. With Theorem 1.9 we may quickly obtain a proof of [3, Theorem 1.2]. Suppose A and B are strongly Morita equivalent; in our notation this means that there is a full Hilbert B -module \mathcal{E} with $A \cong \mathcal{K}(\mathcal{E})$. If A and B have strictly positive elements then \mathcal{E} is countably generated by Corollary 1.5 and we may apply Theorem 1.9 to conclude that $\mathcal{E}^\infty \cong \mathcal{B}^\infty$. Now, as in [8, §2.9],

$$\mathcal{K}(\mathcal{E}^\infty) = \mathcal{K}(\mathcal{E} \otimes \mathcal{K}) \cong \mathcal{K}(\mathcal{E}) \otimes \mathcal{K}(\mathcal{K})$$

and, similarly, $\mathcal{K}(\mathfrak{B}^\infty) \cong \mathcal{K}(\mathfrak{B}) \otimes \mathcal{K}(\mathcal{K})$. Thus

$$A \otimes \mathcal{K} \cong \mathcal{K}(\mathfrak{E}) \otimes \mathcal{K}(\mathcal{K}) \cong \mathcal{K}(\mathfrak{E}^\infty) \cong \mathcal{K}(\mathfrak{B}^\infty) \cong \mathcal{K}(\mathfrak{B}) \otimes \mathcal{K}(\mathcal{K}) \cong B \otimes \mathcal{K}.$$

So A and B are stably isomorphic.

2. Triviality theorems with group actions. Let (A, α, G) be a C^* -dynamical system.

DEFINITION 2.1 (SEE [7, DEFINITION 1]). A Hilbert $(G - A)$ -module \mathfrak{E} is a Hilbert A -module which is also a left G -module satisfying:

- (i) $t \cdot (\xi a) = (t \cdot \xi)\alpha_t(a)$,
- (ii) $t \rightarrow t \cdot \xi$ is continuous,
- (iii) $\langle t \cdot \xi, t \cdot \eta \rangle = \alpha_t(\langle \xi, \eta \rangle)$

for all $\xi, \eta \in \mathfrak{E}, t \in G$ and $a \in A$.

Let \mathfrak{E}_1 and \mathfrak{E}_2 be Hilbert $(G - A)$ -modules. There is an action of G induced on $\mathcal{L}(\mathfrak{E}_1, \mathfrak{E}_2)$, namely $(t \cdot T)(\xi) = t \cdot T(t^{-1} \cdot \xi)$ for $\xi \in \mathfrak{E}_1, T \in \mathcal{L}(\mathfrak{E}_1, \mathfrak{E}_2)$ and $t \in G$. Note that T is G -equivariant iff $t \cdot T = T$ for all $t \in G$. In general, the map $t \rightarrow t \cdot T$ is strongly continuous. T is called G -continuous in case this map is continuous in norm (see [9, 1.3]).

If \mathfrak{E}_1 and \mathfrak{E}_2 are Hilbert $(G - A)$ -modules then we can make $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ into a Hilbert $(G - A)$ -module by defining the G action as follows: $t \cdot (\xi_1, \xi_2) = (t \cdot \xi_1, t \cdot \xi_2)$ for $t \in G, \xi_1 \in \mathfrak{E}_1$, and $\xi_2 \in \mathfrak{E}_2$. Similarly, if \mathfrak{E} is a Hilbert $(G - A)$ -module then so is \mathfrak{E}^∞ . A itself is a Hilbert $(G - A)$ -module where $t \cdot \xi = \alpha_t(\xi)$ for $t \in G$ and $\xi \in A$.

If \mathfrak{E} is a Hilbert $(G - A)$ -module we can make $C_{00}(G, \mathfrak{E})$ (the continuous compactly supported functions from G to \mathfrak{E}) into a pre-Hilbert $(G - A)$ -module as follows:

$$(\xi a)(t) = \xi(t)a, \quad (s \cdot \xi)(t) = s \cdot \xi(s^{-1}t), \quad \langle \xi, \eta \rangle = \int_G \langle \xi(t), \eta(t) \rangle dt$$

for $\xi, \eta \in C_{00}(G, \mathfrak{E}), s \in G$ and $a \in A$.

DEFINITION 2.2 (SEE [9, 1.4]). $L^2(G, \mathfrak{E})$ is the completion of $C_{00}(G, \mathfrak{E})$ as a Hilbert $(G - A)$ -module.

Note that $L^2(G, \mathfrak{E})$ is a completion of the algebraic tensor product $L^2(G) \otimes \mathfrak{E}$ and the G action is the tensor product of the left regular representation with the G action on \mathfrak{E} . In view of [4, 13.11.3], the following result should not be surprising.

LEMMA 2.3. *If \mathfrak{E}_1 and \mathfrak{E}_2 are isomorphic as Hilbert A -modules then $L^2(G, \mathfrak{E}_1)$ and $L^2(G, \mathfrak{E}_2)$ are isomorphic as Hilbert $(G - A)$ -modules (i.e. by a G -equivariant isomorphism of A -modules).*

PROOF. Let U be a unitary operator in $\mathcal{L}(\mathfrak{E}_1, \mathfrak{E}_2)$. Define $V \in \mathcal{L}(L^2(G, \mathfrak{E}_1), L^2(G, \mathfrak{E}_2))$ by $(V\xi)(t) = t \cdot U(t^{-1} \cdot \xi(t))$ for $\xi \in C_{00}(G, \mathfrak{E}_1)$. It is not difficult to check that V is an A -module map, G -equivariant and unitary. Q.E.D.

The Hilbert $(G - A)$ -module version of Theorem 1.9 now follows.

THEOREM 2.4. *Let \mathfrak{E} be a Hilbert $(G - A)$ -module which is countably generated and full as a Hilbert A -module. Then $L^2(G, \mathfrak{E})^\infty$ is isomorphic to $L^2(G, A)^\infty$ by a G -equivariant isomorphism of Hilbert A -modules.*

PROOF. There are obvious G -equivariant isomorphisms $L^2(G, \mathfrak{E})^\infty \simeq L^2(G, \mathfrak{E}^\infty)$ and $L^2(G, A)^\infty \simeq L^2(G, A^\infty)$. By Theorem 1.9 \mathfrak{E}^∞ and A^∞ are isomorphic as Hilbert A -modules and so by Lemma 2.3 there is a G -equivariant isomorphism $L^2(G, \mathfrak{E}^\infty) \simeq L^2(G, A^\infty)$. Q.E.D.

The Hilbert $(G - A)$ -module version of Theorem 1.4 is the following:

THEOREM 2.5 (KASPAROV [9, THEOREM 2.1]). *Let \mathfrak{E} be a Hilbert $(G - A)$ -module which is countably generated as a Hilbert A -module. There is a G -continuous isomorphism from $\mathfrak{E} \oplus L^2(G, A)^\infty$ to $L^2(G, A)^\infty$. If G is compact this isomorphism can be chosen to be G -equivariant.*

PROOF. By Lemma 2.3 and Theorem 1.4 we have equivariant isomorphisms

$$L^2(G, A)^\infty \simeq L^2(G, A^\infty)^\infty \simeq L^2(G, \mathfrak{E} \oplus A^\infty)^\infty \simeq L^2(G, \mathfrak{E})^\infty \oplus L^2(G, A^\infty)^\infty.$$

Let $\phi \in C_{00}(G)$ with $\|\phi\|_2 = 1$. Let $V: \mathfrak{E} \rightarrow L^2(G, \mathfrak{E})$ be given by $(V\xi)(t) = \xi\phi(t)$. It is easy to check that V is a G -continuous isometry. Now define $U: \mathfrak{E} \oplus L^2(G, \mathfrak{E})^\infty \rightarrow L^2(G, \mathfrak{E})^\infty$ by

$$U(\xi_0, \xi_1, \xi_2, \dots) = (V\xi_0 + (1 - VV^*)\xi_1, VV^*\xi_1 + (1 - VV^*)\xi_2, \dots).$$

U defines a G -continuous unitary with

$$U^*(\eta_1, \eta_2, \eta_3, \dots) = (V^*\eta_1, VV^*\eta_2 + (1 - VV^*)\eta_1, VV^*\eta_3 + (1 - VV^*)\eta_2, \dots).$$

Thus

$$\begin{aligned} \mathfrak{E} \oplus L^2(G, A)^\infty &\simeq \mathfrak{E} \oplus L^2(G, \mathfrak{E})^\infty \oplus L^2(G, A^\infty)^\infty \\ &\simeq L^2(G, \mathfrak{E})^\infty \oplus L^2(G, A^\infty)^\infty \simeq L^2(G, A)^\infty. \end{aligned}$$

The resulting isomorphism is G -continuous.

If G is compact we may take $\phi = 1$. Then V and U are equivariant and thus $\mathfrak{E} \oplus L^2(G, A)^\infty \simeq L^2(G, A)^\infty$ by a G -equivariant unitary. Q.E.D.

To conclude we shall explain why Theorem 2.4 is the equivariant version of the triviviality theorem of [2]. The equivariant version of [2, Lemma 2.5] is

COROLLARY 2.6. *Let (A, α, G) be a C^* -dynamical system and suppose A has a strictly positive element. If p in $M(A)$ is a full invariant projection then $p \otimes 1 \sim 1 \otimes 1$ in $M(A \otimes \mathfrak{K}(L^2(G)^\infty))$ by an invariant partial isometry.*

Theorem 1.9 and Corollary 2.6 (in the case of a trivial action) are, in fact, proving the same thing. Indeed, suppose B has a strictly positive element and \mathfrak{E} is a countably generated full Hilbert B -module. Let $A = \mathfrak{K}(\mathfrak{E} \oplus B)$; A is the linking algebra for the strongly Morita equivalent C^* -algebras $\mathfrak{K}(\mathfrak{E})$ and B as in [2, Theorem 2.8 and 3, Theorem 1.1]. By Lemma 1.2, B , when considered as a Hilbert B -module, is countably generated (by a single element in fact). Thus $\mathfrak{E} \oplus B$ is countably generated as a Hilbert B -module; so by Corollary 1.5, $A = \mathfrak{K}(\mathfrak{E} \oplus B)$ has a strictly positive element. Let p and q be the projections in $M(A)$ with ranges \mathfrak{E} and B , respectively. It is easy to check that AqA is dense in A and similarly ApA is dense in A because \mathfrak{E} is full. Thus p and q are full projections [2, Lemma 1.1].

Now as in [8, §2.9] $\mathcal{K}(\mathcal{E} \oplus B) \otimes \mathcal{K} \cong \mathcal{K}((\mathcal{E} \oplus B) \otimes \mathcal{K})$, so
 $A \otimes \mathcal{K} \cong \mathcal{K}(\mathcal{E} \otimes \mathcal{K} \oplus B \otimes \mathcal{K})$.

Under this isomorphism $p \otimes 1$ and $q \otimes 1$ become the projections onto $\mathcal{E} \otimes \mathcal{K} \cong \mathcal{E}^\infty$ and $B \otimes \mathcal{K} \cong B^\infty$, respectively. Thus $p \otimes 1 \sim 1 \otimes 1 \sim q \otimes 1$ gives $\mathcal{E}^\infty \cong B^\infty$.

PROOF OF COROLLARY 2.6. Let $\mathcal{E} = pA$; then \mathcal{E} is a Hilbert $(G - A)$ -module. As $\mathcal{K}(\mathcal{E}) = pAp$, which being a corner of A has a strictly positive element [3, Proposition 2.3], we have that \mathcal{E} is countably generated by Corollary 1.5. Also, as $\langle \mathcal{E}, \mathcal{E} \rangle = \overline{pAp}$ we see that \mathcal{E} is full. So $\mathcal{E} \otimes L^2(G)^\infty \cong A \otimes L^2(G)^\infty$ by an equivariant isomorphism, that is, $p \otimes 1 \sim 1 \otimes 1$ in

$$\mathcal{L}(A \otimes L^2(G)^\infty) \cong M(A \otimes \mathcal{K}(L^2(G)^\infty))$$

by an invariant partial isometry. Q.E.D.

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