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A PROPERTY OF COMPACT OPERATORS

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ABSTRACT. In this note it is shown that if T is a compact linear operator on a wide class of Banach spaces of the form $C(S)$, compact S, or $L^1(S, \Sigma, \mu)$, then $\|I + T\| = 1 + \|T\|$. This generalizes similar theorems for the spaces $C[0, 1]$ and $L^1(0, 1)$.

In [**1**] and [**2**] it was shown that if T is any compact linear operator on either of the Banach spaces $L^1(0, 1)$ or $C[0, 1]$, then T satisfies $\|I + T\| = 1 + \|T\|$, where I is the identity operator. In both papers the method used was to prove this for operators with finite-dimensional range, and from this deduce the theorem for arbitrary compact operators. In this note we use characterizations of compact operators on $C(S)$ or $L^1(S, \Sigma, \mu)$ to prove similar theorems for general Banach spaces of these forms.

1. $C(S)$.

THEOREM A. *Let S be a compact Hausdorff space and $C(S)$ be the supnorm Banach space of continuous complex valued functions on S. Then every compact linear operator T on $C(S)$ satisfies $\|I + T\| = 1 + \|T\|$ if and only if S is a perfect set, i.e. S has no isolated points.*

PROOF. Before beginning the proof we note the following.

I. Suppose $f \in C(S)$ with $\|f\| = 1$. Then two applications of Urysohn's Lemma gives that to each open set U and $p \in U$ there exists $F \in C(S)$ with $\|F\| = 1$, $F(p) = 1$ and $F(t) = f(t)$ for $t \notin U$.

II. Let $s_0 \in S$ and suppose μ is a measure in the dual space of $C(S)$. Since such measures are regular it follows that to each $p \in S$ and $\varepsilon > 0$ there exists an open set U with $p \in U$ and $|\mu(U \setminus \{p\})| < \varepsilon$ [**3**, p. 137].

III. If T: $C(S) \to C(S)$ is a compact operator then there exists a family $\{d\mu_s\}_{s \in S}$ of measures on S such that for each $f \in C(S)$ and $s \in S$, $Tf(s) = \int_S f(t)\, d\mu_s(t)$ and further the map $s \to d\mu_s$ is continuous in the norm topology on $C(S)^*$ [**3**, p. 490]. As a result, if $\varepsilon > 0$ and $s_0 \in S$, there exists a neighborhood U of s_0 such that $s \in U$ implies $\|d\mu_s - d\mu_{s_0}\| < \varepsilon$.

Now suppose S is a compact Hausdorff space with no isolated points and let T be a compact linear operator on $C(S)$. Suppose further that T has the form $Tf(s) = \int_S f(t)\, d\mu_s(t)$ for $s \in S$ as described in III. Let $g \in C(S)$ with $\|g\| = 1$ and

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$\|Tg\| = \|T\|$. Then

$$\|T\| = \sup_s \left| \int_S g(t)\, d\mu_s(t) \right| = \left| \int_S g(t)\, d\mu_{s_0}(t) \right|$$

for some $s_0 \in S$. By multiplying g by the appropriate constant of modulus 1 we may assume that $\int_S g(t)\, d\mu_{s_0}(t) > 0$ so that $\|T\| = \int_S g(t)\, d\mu_{s_0}(t)$.

Let $\varepsilon > 0$. Then by II and III there exists an open set U such that $s_0 \in U$, $|\mu_{s_0}(U \setminus \{s_0\})| < \varepsilon/4$ and if $s \in U$, then $\|d\mu_s - d\mu_{s_0}\| < \varepsilon/2$. Since s_0 is not an isolated point of S, there exists $s_1 \in U$, $s_1 \neq s_0$, and by I there exists a function $F \in C(S)$ with $F(s_1) = 1$, $\|F\| = 1$, $F(s_0) = g(s_0)$ and $F(t) = g(t)$ for $t \notin U$. Then

$$\|I + T\| \geqslant \|(I + T)F\| \geqslant |(I + T)F(s_1)| = \left| F(s_1) + \int_S F(t)\, d\mu_{s_1}(t) \right|.$$

Now

$$F(s_1) + \int_S F(t)\, d\mu_{s_1}(t) = F(s_1) + \int_S F(t)\big[d\mu_{s_1}(t) - d\mu_{s_0}(t)\big] + \int_S [F(t) - g(t)]\, d\mu_{s_0}(t) + \int_S g(t)\, d\mu_{s_0}(t).$$

But $F(s_1) = 1$ and $\int_S g(t)\, d\mu_{s_0}(t) = \|T\|$, while

$$\left| \int_S F(t)\big[d\mu_{s_1}(t) - d\mu_{s_0}(t)\big] \right| \leqslant \|d\mu_{s_1} - d\mu_{s_0}\| < \frac{\varepsilon}{2}$$

and

$$\left| \int_S [F(t) - g(t)]\, d\mu_{s_0}(t) \right| = \left| \int_{U \setminus \{s_0\}} [F(t) - g(t)]\, d\mu_{s_0}(t) \right| < 2|\mu_{s_0}(U \setminus \{s_0\})| < \frac{\varepsilon}{2}.$$

Therefore

$$\|I + T\| \geqslant \|(I + T)F\| \geqslant |(I + T)F(s_1)| = \left| F(s_1) + \int_S F(t)\, d\mu_{s_1}(t) \right| \geqslant 1 + \|T\| - \varepsilon.$$

This holds for all $\varepsilon > 0$. Therefore $\|I + T\| \geqslant 1 + \|T\|$. But $\|I + T\| \leqslant 1 + \|T\|$ always. Thus if S has no isolated points then $\|I + T\| = 1 + \|T\|$ for all compact operators T on $C(S)$.

Conversely, suppose S has an isolated point s_0. Define $e_{s_0}\colon S \to \mathbf{C}$ (complex numbers) by $e_{s_0}(s_0) = 1$ and $e_{s_0}(s) = 0$ for $s \neq s_0$. Then $e_{s_0} \in C(S)$. If T is defined on $C(S)$ by $Tf(s) = -f(s_0)e_{s_0}$ for $f \in C(S)$, then T has one-dimensional range and is therefore a compact operator on $C(S)$. Further if $f \in C(S)$, then $(I + T)f(s) = f(s)$ for $s \neq s_0$ and $(I + T)f(s_0) = 0$. Therefore $\|I + T\| = 1 < 1 + \|T\|$. Thus if a compact set S has an isolated point then there exists a compact operator T on $C(S)$ with $\|I + T\| < 1 + \|T\|$, and the proof is complete.

2. $L^1(S, \Sigma, \mu)$. Our results in this case are not as complete as for $C(S)$. Here we consider positive σ-finite measure spaces (S, Σ, μ) which satisfy the following additional property which we call Property (V).

Property (V). A positive σ-finite measure space (S, Σ, μ) is said to satisfy Property (V) if to each $s_0 \in S$ there exists a decreasing family $\{E_n(s_0)\}$ in Σ satisfying

(i) $s_0 \in E_n(s_0)$,

(ii) $\mu(E_n(s_0)) > 0$,

and such that if f is an integrable function on S with values in a Banach space, then for almost all $s_0 \in S$,

$$\lim_{n \to \infty} \frac{1}{\mu(E_n(s_0))} \int_{E_n(s_0)} f(s)\, d\mu(s) = f(s_0).$$

The convergence is convergence in the Banach space norm. Further if $\mu(\{s_0\}) = 0$, then $\{E_n(s_0)\}$ may be chosen to satisfy $\lim_{n \to \infty} \mu(E_n(s_0)) = 0$.

Property (V) holds for a wide class of measures including Lebesgue measure on R^n, or on cubes in R^n. Indeed, it holds for arbitrary complete regular measures on these sets. Property (V) is related to the question of when certain Vitali conditions hold [**3**, p. 217; **4**, p. 187; **5**, p. 209].

We now prove a theorem similar to Theorem A for the Banach spaces $L^1(S, \Sigma, \mu)$ where (S, Σ, μ) is a positive σ-finite measure space satisfying Property (V). We first consider the case when the measure μ has an atom.

PROPOSITION. *If (S, Σ, μ) is a positive σ-finite measure space and if $\mu(\{s_0\}) \neq 0$ for some $s_0 \in S$, then there exists a compact linear operator T on $L^1(S, \Sigma, \mu)$ satisfying $\|I + T\| < 1 + \|T\|$.*

PROOF. Let $s_0 \in S$ with $\mu(\{s_0\}) \neq 0$ and again let $e_{s_0}(s) = 1$ if $s = s_0$ and $e_{s_0}(s) = 0$ otherwise. Then $e_{s_0} \in L^1(S, \Sigma, \mu)$ and if $Tf(s) = -f(s_0)e_{s_0}$ for $f \in L^1(S, \Sigma, \mu)$, then T is a well-defined compact linear operator on $L^1(S, \Sigma, \mu)$. Since $(I + T)f(s) = 0$ if $s = s_0$ and $(I + T)f(s) = f(s)$ for $s \neq s_0$, it follows that

$$\|(I + T)f\| = \int_{S \setminus \{s_0\}} |f(s)|\, d\mu(s) \leq \int_S |f(s)|\, d\mu(s) = \|f\|.$$

Hence $\|I + T\| = 1 < 1 + \|T\|$.

As a converse we have

THEOREM B. *Let (S, Σ, μ) be a positive σ-finite measure space satisfying Property* (V) *and suppose further that for each $s \in S$, $\{s\}$ is measurable and $\mu(\{s\}) = 0$. If T is a compact linear operator on $L^1(S, \Sigma, \mu)$ then $\|I + T\| = 1 + \|T\|$.*

PROOF. Let T be a compact operator on $L^1(S, \Sigma, \mu)$. Then there exists a $\mu \times \mu$-measurable function $K\colon S \times S \to \mathbf{C}$ such that $Tf(t) = \int_S K(s, t) f(s)\, d\mu(s)$. Further,

(i) $$\|T\| = \operatorname*{ess\,sup}_s \int_S |K(s, t)|\, d\mu(t)$$

and

(ii) there exists a set E of measure 0 and a norm compact subset $\mathcal{K} \subset L^1(S, \Sigma, \mu)$ such that if K_s is defined by $K_s(t) \equiv K(s, t)$ for $(s, t) \in S \times S$, then $K_s \in \mathcal{K}$ for all $s \notin E$ [**3**, p. 507].

Let $\varepsilon > 0$. Since $\|T\| = \operatorname{ess\,sup}_s \int_S |K(s, t)|\, d\mu(t)$, there exists $s_0 \in S$ such that $s_0 \notin E$ and $\int_S |K(s_0, t)|\, d\mu(t) \geqslant \|T\| - \varepsilon$. Also since (S, Σ, μ) satisfies Property (V) there exists a decreasing family of measurable sets $\{E_n(s_0)\} \equiv \{E_n\}$ such that $\mu(E_n) > 0$, $\lim_{n\to\infty} \mu(E_n) = 0$ and the vector valued function $s \to K_s$ satisfies

$$\lim_{n\to\infty} \frac{1}{\mu(E_n)} \int_{E_n} K_s \, d\mu(s) = K_{s_0}$$

in norm.

Suppose such $s_0 \in S$ and $\{E_n\}$ are chosen. Then for each positive integer n let $g_n(s) = 1/\mu(E_n)$ if $s \in E_n$ and $g_n(s) = 0$ otherwise. Clearly, $g_n \in L^1(S, \Sigma, \mu)$ and $\|g\| = 1$. Further

$$\|(I + T)g_n\| = \int_S \left| g_n(t) + \int_S K(s, t) g_n(s)\, d\mu(s) \right| d\mu(t).$$

For each positive integer n, we have

$$\begin{aligned}
&\int_S \left| g_n(t) + \int_S K(s, t) g_n(s)\, d\mu(s) \right| d\mu(t) \\
&\quad = \int_{E_n} \left| \frac{1}{\mu(E_n)} + \int_{E_n} \frac{1}{\mu(E_n)} K(s, t)\, d\mu(s) \right| d\mu(t) \\
&\qquad + \int_{S \setminus E_n} \left| \int_{E_n} \frac{1}{\mu(E_n)} K(s, t)\, d\mu(s) \right| d\mu(t) \\
&\quad \geqslant \int_{E_n} \frac{1}{\mu(E_n)} d\mu(t) - \int_{E_n} \left| \int_{E_n} \frac{1}{\mu(E_n)} K(s, t)\, d\mu(s) \right| d\mu(t) \\
&\qquad + \int_{S \setminus E_n} \left| \int_{E_n} \frac{1}{\mu(E_n)} K(s, t)\, d\mu(s) \right| d\mu(t) \\
&\quad = 1 + \int_S \left| \int_{E_n} \frac{K(s,t)}{\mu(E_n)} d\mu(s) \right| d\mu(t) - 2 \int_{E_n} \left| \int_{E_n} \frac{K(s,t)}{\mu(E_n)} d\mu(s) \right| d\mu(t).
\end{aligned}$$

By our choice of s_0 we have

$$\lim_{n\to\infty} \frac{1}{\mu(E_n)} \int_{E_n} K_s \, d\mu(s) = K_{s_0}$$

in norm so that

$$(*) \qquad \lim_{n\to\infty} \int_S \left| \int_{E_n} \frac{K(s,t)}{\mu(E_n)} d\mu(s) \right| d\mu(t) = \int_S |K(s_0, t)| d\mu(t) \geqslant \|T\| - \varepsilon.$$

It remains to be shown that

$$\int_{E_n}\left|\int_{E_n}\frac{K(s,t)}{\mu(E_n)}d\mu(s)\right|d\mu(t)$$

can be made arbitrarily small.

Let E be the set of measure 0 and $\mathcal{K}$ be the norm compact subset of $L^1(S, \Sigma, \mu)$ for which $K_s \in \mathcal{K}$ for all $s \notin E$ as described at the beginning of the proof. Let $\mathcal{K}_0$ = norm closure of $\{K_s | s \notin E\}$ in $L^1(S, \Sigma, \mu)$. Then $\mathcal{K}_0$ is compact in the norm topology.

Again consider $\{E_n\} = \{E_n(s_0)\}$. For each positive integer n, define Φ_n on $\mathcal{K}_0$ by $\Phi_n(f) = \int_{E_n}|f|\,d\mu = \int_{E_n}|f(t)|\,d\mu(t)$ for $f \in \mathcal{K}_0$. Since Φ_n is a real valued continuous function on the compact set $\mathcal{K}_0$, Φ_n attains a maximum γ_n, say where $\gamma_n \geqslant 0$. That is, $\gamma_n = \max_{f\in\mathcal{K}_0}\int_{E_n}|f|\,d\mu$. Further, since $E_n \supset E_{n+1}$, we have $\gamma_n \geqslant \gamma_{n+1}$. Let $\gamma = \lim_{n\to\infty}\gamma_n$. We want to show that $\gamma = 0$. If $\gamma_n = 0$ for some integer n then $\gamma = 0$. So assume $\gamma_n > 0$ for all n and let $\mathcal{K}_n$ = norm closure of

$$\left\{ K_s | s \notin E \text{ and } \int_{E_n}|K_s(t)|d\mu(t) > \frac{\gamma}{2}\right\}.$$

Since $\mathcal{K}_n \subset \mathcal{K}_0$ each $\mathcal{K}_n$ in compact in the norm topology of $L^1(S, \Sigma, \mu)$. Also $\mathcal{K}_{n+1} \subset \mathcal{K}_n$ since $E_{n+1} \subset E_n$. Further each $\mathcal{K}_n$ is nonempty. Therefore $\{\mathcal{K}_n\}$ is a nested sequence of nonempty compact subsets of $L^1(S, \Sigma, \mu)$ whence $\cap\mathcal{K}_n \neq \varnothing$. Let $F \in \cap\mathcal{K}_n$. Then $\int_{E_n}|F(t)|\,d\mu(t) \geqslant \gamma/2$ for all n. But $\lim_{n\to\infty}\mu(E_n) = 0$. Therefore $\gamma = 0$ as claimed.

Now $\{K_s | s \notin E\}$ is dense in $\mathcal{K}_0$ so that

$$\gamma_n = \sup_{s\notin E}\int_{E_n}|K(s, t)|d\mu(t).$$

Since $\gamma_n \downarrow 0$ it follows that

$$\sup_{s\notin E}\int_{E_n}|K(s, t)|d\mu(t) < \varepsilon$$

for large n, and so $\int_{E_n}|K(s, t)|\,d\mu(t) < \varepsilon$ for large n and all $s \notin E$. Therefore

$$\begin{aligned}\int_{E_n}\left|\int_{E_n}\frac{K(s,t)}{\mu(E_n)}d\mu(s)\right|d\mu(t) &\leqslant \int_{E_n}\left[\int_{E_n\setminus E}\frac{|K(s,t)|}{\mu(E_n)}d\mu(s)\right]d\mu(t)\\ &= \int_{E_n\setminus E}\left[\int_{E_n}\frac{|K(s,t)|}{\mu(E_n)}d\mu(t)\right]d\mu(s)\\ &< \int_{E_n\setminus E}\frac{\varepsilon}{\mu(E_n)}d\mu(s) \quad \text{for large } n,\end{aligned}$$

the equality holding by Tonelli's Theorem. Thus

$$\int_{E_n}\left|\int_{E_n}\frac{K(s,t)}{\mu(E_n)}d\mu(s)\right|d\mu(t) < \frac{\varepsilon\mu(E_n\setminus E)}{\mu(E_n)} = \varepsilon$$

for large n. Combining this with $(*)$ we have

$$(**) \qquad 1 + \int_S \left| \int_{E_n} \frac{K(s,t)}{\mu(E_n)} d\mu(s) \right| d\mu(t) - 2\int_{E_n} \left| \int_{E_n} \frac{K(s,t)}{\mu(E_n)} d\mu(s) \right| d\mu(t)$$
$$> 1 + \|T\| - \varepsilon - 2\varepsilon = 1 + \|T\| - 3\varepsilon$$

for large n. Since ε is an arbitrary positive number and $\|(I + T)g_n\| >$ left-hand side of $(**)$ for each n, we have $\|I + T\| \geqslant 1 + \|T\|$. Finally, since $\|I + T\| \leqslant 1 + \|T\|$ always, we conclude that if T is a compact linear operator on $L^1(S, \Sigma, \mu)$ where (S, Σ, μ) is a positive γ-finite measure space which satisfies Property (V) and $\mu(\{s_0\}) = 0$ for all $s_0 \in S$, then $\|I + T\| = 1 + \|T\|$ as required.

3. We conclude with a few general remarks.

(a) If B is finite dimensional, then there exists a compact operator T for which $\|I + T\| < 1 + \|T\|$. Indeed, every linear operator on B is compact and choosing $T = -I$ gives the required example.

(b) If B is Hilbert space and $\{e_n\}$ is an orthonormal basis then $T: \sum_{n=1}^{\infty} a_n e_n \to -a_1 e_1$ is a compact operator for which $\|I + T\| = 1 < 1 + \|T\|$.

(c) If $\|I + S\| = 1 + \|S\|$ holds for every compact operator S in the dual space B^* of a Banach space B, then $\|I + T\| = 1 + \|T\|$ for every compact operator in B. Indeed, if T is a compact operator on B, then T^* is compact on B^*. But $\|I^* + T^*\| = 1 + \|T^*\|$. Hence $\|I + T\| = \|I^* + T^*\| = 1 + \|T^*\| = 1 + \|T\|$ as claimed.

(d) However, even if $\|I + T\| = 1 + \|T\|$ for all compact operators T on a Banach space B, this property may fail for B^*. As an example let $B = C[0, 1]$. Then $B^* = BV[0, 1]$. Then $B^* = BV[0, 1]$. Then $\|I + T\| = 1 + \|T\|$ for all compact T on B; but if we define S on $BV[0, 1]$ by $S = -j_0(\mu)\nu_0$ where $j_0(\mu)$ is the jump of μ at 0 and ν_0 is evaluation at 0, i.e. $\int_0^1 f(t)\,d\nu_0(t) = f(0)$ for all $f \in C[0, 1]$. Then using the by now usual argument, $\|I + S\| = 1 < 1 + \|S\|$.

Bibliography

1. V. F. Babinko and S. A. Picugov, *On a property of compact operators on the space of integrable functions*, Ukrain Mat. Z. **33** (1981), 491–492.

2. I. K. Daugavet, *A property of completely continuous functions on the space C*, Uspehi Mat. Nauk **18** (1963), 157–158.

3. N. Dunford and J. T. Schwartz, *Linear operators*. Part I, Interscience, New York, 1958.

4. C. A. Hayes and C. Y. Pauc, *Derivation and martingales*, Springer-Verlag, Berlin and New York, 1970.

5. G. E. Shilov and B. L. Gurevich, *Integral, measure and derivation: a unified approach* (translated by R. A. Silverman), Prentice-Hall, Englewood Cliffs, N.J., 1966.

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