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NEWTON'S METHOD AND SYMBOLIC DYNAMICS

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ABSTRACT. By the use of symbolic dynamics, this note proves a result of B. Barna concerning real polynomials of degree at least 4 and having all distinct simple real roots. Specifically, the set of initial points, for which Newton's method fails to converge to a root of the given polynomial, is homeomorphic to a Cantor set. Also the note shows that the requirement for simple roots may be relaxed, and one still has Barna's result being valid.

Introduction. In [**9**], S. Smale states that B. Barna proves that for a polynomial with all real roots, Newton's method converges to a root starting with almost every real number and the exceptional set of initial points is homeomorphic to a Cantor set. Specifically, Barna [**1**] proves that for a polynomial of at least degree 4 with distinct simple real roots, the exceptional set of initial points is homeomorphic to a Cantor set. In this note, Barna's result is proven with the use of symbolic dynamics. Moreover, one can extend Barna's result to polynomials having at least 4 distinct real roots and only real roots.

Besides Smale's paper, at least two papers that discuss Newton's method have been formulated. The papers are by D. Saari and J. Urenko [**8**] and by M. Hurley and C. Martin [**7**].

I want to express my thanks to M. Shub for all the valuable discussions regarding the dynamics of Newton's method and to the referee for his valuable comments on this note.

Preliminaries. Let $f(x) = \prod_{i=0}^{n-1}(x - r_i)$ where $r_i \neq r_j$ $(i \neq j)$, $r_i \in \mathbf{R}$ for $i = 0, 1, \ldots, n-1$ and $n > 4$. Define $T\colon \mathbf{R} \to \mathbf{R}$ as follows:

$$Tx = x - f(x)/f'(x)$$

where f' is the derivative of f and call T the Newton transform of f.

If $\mathfrak{S} = \{x \in \mathbf{R}\colon T^m x \to r_i \text{ as } m \to \infty \text{ for some } i = 0, \ldots, n-1\}$, then let $\mathfrak{E} = \mathbf{R} \setminus \mathfrak{S}$.

BARNA'S THEOREM [**1**]. *If f, T, $\mathfrak{E}$ are as above, then $\mathfrak{E}$ is homeomorphic to a Cantor set (excluding a countable set).*

Observations. (1) $T'x = f(x)f''(x)/[f'(x)]^2$ and consequently the critical numbers of T (i.e., those values of x for which $T'x = 0$) are the roots of f, $r_0, r_1, \ldots, r_{n-1}$, and the points of inflection of f at $s_1, s_2, \ldots, s_{n-2}$, say. (For f, there are exactly

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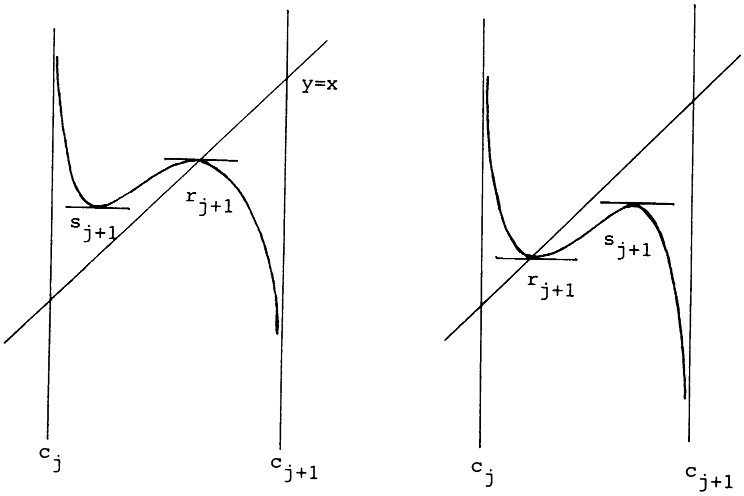

FIGURE 1a. T between c_j and c_{j+1}

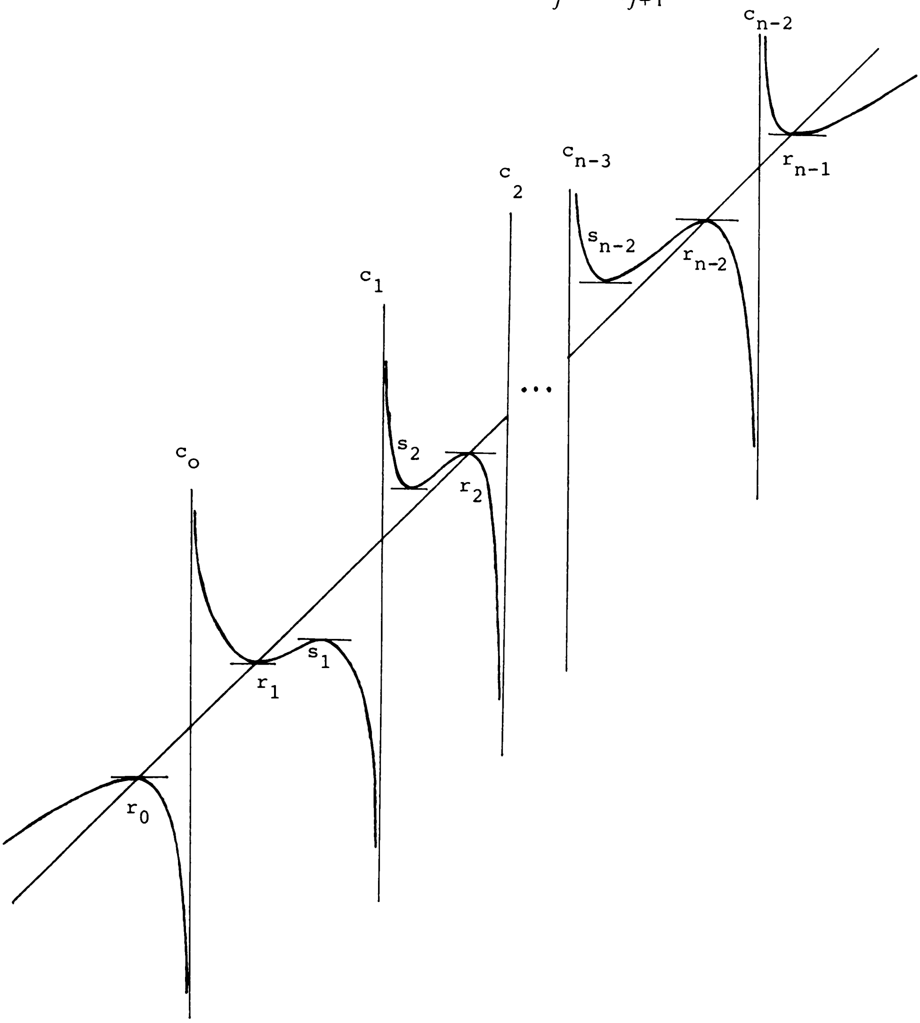

FIGURE 1b. T with all real roots each of multiplicity 1

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$(n-1)$ critical numbers $c_0, c_1, \ldots, c_{n-2}$, and since there is only one value of x between two consecutive critical numbers of f that determines a point of inflection, there are exactly $(n-2)$ points of inflection.)

(2) From T', one concludes that between c_j and c_{j+1}, T is monotone decreasing from c_j to r_{j+1} or s_{j+1} (whichever occurs first), then is monotone increasing to s_{j+1} or r_{j+1} (whichever occurs second), and finally is monotone decreasing to c_{j+1} (see Figure 1). For $x < c_0$, T is monotone increasing to r_0 and then monotone decreasing to c_0. For $x > c_{n-2}$, T is monotone decreasing to r_{n-1} and then monotone increasing.

Symbolic dynamics. To derive the symbolic dynamics, some notation is necessary. Let $t_i = \min(r_i, s_i)$ and $u_i = \max(r_i, s_i)$ for $i = 1, \ldots, n-2$. Partition $\mathbf{R}$ into the following intervals:

(1) $C_0 = (-\infty, c_0)$, $C_{n-1} = (c_{n-2}, \infty)$;

(2) $C_i = [t_i, u_i]$ for $i = 1, \ldots, n-2$;

(3) $L_i = (c_{i-1}, t_i)$, $R_i = (u_i, c_i)$ for $i = 1, \ldots, n-2$.

(One notices that $r_0 \in C_0$, $r_{n-1} \in C_{n-1}$, and $r_i \in C_i$ for $i = 1, \ldots, n-2$. Also $c_0, c_1, \ldots, c_{n-2}$ do not belong to any of the elements of the partition.)

If

$$\begin{aligned}\mathcal{P} &= \{C_0, C_1, \ldots, C_{n-1}, L_1, R_1, L_2, R_2, \ldots, L_{n-2}, R_{n-2}\}\\ &= \{I_0, I_1, \ldots, I_{n-1}, I_n, I_{n+1}, I_{n+2}, I_{n+3}, \ldots, I_{3n-6}, I_{3n-5}\},\end{aligned}$$

define A a matrix of zeros and ones as follows: For $I_i, I_j \in \mathcal{P}$,

$$a_{ij} = \begin{cases} 1 & \text{if } I_i \cap T^{-1}I_j \neq \varnothing, \\ 0 & \text{if } I_i \cap T^{-1}I_j = \varnothing. \end{cases}$$

Then

$$A = \left[\begin{array}{c|c} I & W \\ \hline Z & V \end{array}\right]$$

where I is the $n \times n$ identity matrix; Z is the $(2n-6) \times n$ zero matrix, and W is an $n \times (2n-6)$ matrix of zeros and ones. V is a $(2n-6) \times (2n-6)$ matrix built from the 2×2 matrices

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

V can be interpreted as an $(n-3) \times (n-3)$ "matrix" of matrices:

(1) $V_{ii} = J$ for $i = 1, \ldots, n-3$;

(2) for $1 \leqslant i \leqslant n-4$, $V_{ki} = M$ for $k > i$;

(3) for $2 \leqslant i \leqslant n-3$, $V_{ki} = N$ for $k < i$.

For example if $n = 6$, then V has the form

$$\begin{bmatrix} J & N & N \\ M & J & N \\ M & M & J \end{bmatrix}.$$

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One can associate with all but a countable number of x's a one-sided sequence of integers from the set $\{0, 1, 2, \ldots, 3n-5\}$:

For $x \notin \mathcal{C}$, a certain countable set of real numbers, consider the forward orbit of x, $\{x, Tx, T^2x, \ldots\}$. Each T^kx belongs to some $I_{x_k} \in \mathcal{P}$. To such an x, one assigns the sequence $\mathbf{x} = x_0x_1x_2x_3 \cdots$ where $x \in T^{-k}I_{x_k}$. One should note that for Tx, one has the sequence $\sigma(\mathbf{x}) = x_1x_2x_3x_4 \cdots$, i.e., the action of T is reflected by the one-sided shift σ. Moreover, the set of all sequences which can occur is exactly

$$X = \{\mathbf{x} = x_0x_1x_2 \cdots : a_{x_ix_{i+1}} = 1 \text{ for all } i \geqslant 0\}$$

with $a_{x_ix_{i+1}}$ refers to the entries of A called the transition matrix.

As pointed out above, the only real numbers which do not belong to any member of $\mathcal{P}$ are $c_0, c_1, \ldots, c_{n-2}$. Consequently, the countable set of omitted real numbers is

$$\mathcal{C} = \{x \in \mathbf{R}\colon T^kx \to c_i \text{ for some } i = 0, 1, \ldots, n-2\}.$$

Because the matrix A captures the orbit structure of T, one can make an interpretation of the submatrices of A, particularly I and V. The $n \times n$ identity matrix I indicates that if $T^kx \in C_i$ for some k and some i, the forward orbit of x remains in that particular C_i and converges to r_i. The matrix V captures the behavior of those x, not in $\mathcal{C}$, for which the forward orbit does not converge to a root of f, i.e., Newton's method fails. Using the identities $J^2 = I = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, $M^2 = JM = NJ = NM = M$ and $N^2 = JN = MJ = MN = N$, one concludes that V^2 as a $(2n-6) \times (2n-6)$ matrix has only two zero entries

$$\sum_{j=1}^{n-3} V_{1j}V_{j1} = J^2 + (n-4)NM = I + (n-4)M = \begin{bmatrix} n-3 & 0 \\ n-4 & 1 \end{bmatrix},$$

$$\sum_{j=1}^{n-3} V_{(n-3)j}V_{j(n-3)} = J^2 + (n-4)MN = I + (n-4)N = \begin{bmatrix} 1 & n-3 \\ 0 & n-4 \end{bmatrix}.$$

Therefore V^4 has no zeros at all, and V is an irreducible matrix (see [**5**]). Hence for any positive integer m, there exist orbits of period m, i.e., for some $x \notin \mathcal{C}$, $T^mx = x$ but $T^kx \neq x$ for $0 < k \leqslant m-1$. With the aid of a few propositions, one can use the sequences determined by V to conclude that $\mathcal{E} \setminus \mathcal{C}$ is totally disconnected and perfect, or homeomorphic to a Cantor set.

Propositions. In [**2**], Barna proves that the equation $T'x = -a$ for $a > 0$ has exactly one solution in each L_i and R_i: Let $a > 0$. Since $T'x \to -\infty$ as $x \to c_i^-$ or as $x \to c_{i+1}^+$ for $i = 0, 1, \ldots, n-2$, and since $T'z = 0$ for $z \in \{r_0, r_1, \ldots, r_{n-1}, s_1, \ldots, s_{n-2}\}$, $T'x = -a$ has at least $2(n-1)$ solutions. However, because $T'x = -a$ is equivalent to $f(x)f''(x) + a[f'(x)]^2 = 0$ which has at most $2(n-1)$ solutions, $T'x = -a$ for $a > 0$ has exactly one solution for each L_i and R_i.

For A, in particular from V, one observes that, within each interval (c_{i-1}, c_i) for $i = 1, \ldots, n-2$, there exist points of period two, i.e., $T^2x = x$ but $Tx \neq x$, because there are x's belonging to $L_i \cap T^{-1}R_i \cap T^{-2}L_i \cap T^{-3}R_i \cap \cdots$ for $i = 1, \ldots, n-2$. ($L_i \cap T^{-1}R_i \cap T^{-2}L_i \cap T^{-3}R_i \cap \cdots$ is nonempty since $L_i \cap T^{-1}R_i$ is properly contained in L_i in the sense that

$$\overline{(L_i \cap T^{-1}R_i)} \cap \{c_{i-1}, t_i\} = \varnothing;$$

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otherwise it would be possible for

$$\overline{\left(L_i \cap T^{-1}R_i \cap T^{-2}L_i \cap \cdots\right)} \neq \varnothing$$

but $L_i \cap T^{-1}R_i \cap T^{-2}L_i \cap \cdots = \varnothing$.)

PROPOSITION 1. *For each $i = 1, \ldots, n-2$, the interval (c_{i-1}, c_i) contains exactly one period-two cycle.*

PROOF. Suppose $c_{i-1} < x < y < t_i$ with $T'x > T'y$. As $x \to c_{i-1}^+$, $T'x \to -\infty$. Thus, there is a $z \in (c_{i-1}, x)$ such that $T'z = T'y$, a contradiction to the fact that $T'x = -a$ for $a > 0$ has but one solution in L_i. If $T'x = T'y$, then again one contradicts the same fact. Consequently, T' is strictly increasing on L_i. Similarly, one concludes that T' is strictly decreasing on R_i. As stated before, (c_{i-1}, c_i) has at least one period-two cycle. If $\{x_1, Tx_1\}$ and $\{x_2, Tx_2\}$ are two distinct period-two cycles of (c_{i-1}, c_i) and $x_1 < x_2$, then $x_1 < x_2 < t_i$ and $u_i < Tx_2 < Tx_1$. Because $T^2([x_1, x_2]) = [x_1, x_2]$, there is an $e \in (x_1, x_2)$ with $(T^2)'e = 1$. By the monotonicity of T', $e < x_2$ implies that $1 = (T^2)'e > (T^2)'x_2$. Moreover, since $(T^2)'r_i = 0$ and $x_2 = T^2x_2 < T^2r_i = r_i$, there exists a $z \in (x_2, r_i)$ such that $T^2z = z$ and $(T^2)'z \geqslant 1$. Because r_i is the only fixed point of T in (c_{i-1}, c_i), $Tz \neq z$ and thus $z \in L_i$. Again by the monotonicity of T', $x_2 < z$ implies that $1 > (T^2)'x_2 > (T^2)'z \geqslant 1$, a contradiction. Thus (c_{i-1}, c_i) contains exactly one period-two cycle. □

PROPOSITION 2. *If $\{\alpha, \beta\}$ is the unique period-two cycle of (c_{i-1}, c_i), then $(T^2)'\alpha = (T^2)'\beta > 1$.*

PROOF. Because $\{\alpha, \beta\}$ is a cycle of period-two, $(T^2)'\alpha = (T^2)'\beta$. Suppose $(T^2)'\alpha < 1$. Then there is an open interval $K \subset (\alpha, \beta)$ such that $\alpha \in \overline{K}$ and $T^2x > x$ for $x \in K$. Because $T^2r_i = r_i$ and $(T^2)'r_i = 0$, there is an open interval $H \subset (\alpha, r_i)$ such that $r_i \in \overline{H}$ and $T^2x < x$ for $x \in H$. For $x \in (\alpha, r_i)$, $T^2x - x < 0$ if $x \in K$ and $T^2x - x > 0$ if $x \in H$. Thus there is a $z \in (\alpha, r_i)$ for which $T^2z - z = 0$, i.e., $\{z, Tz\}$ is a period-two cycle, a contradiction. Suppose $(T^2)'\alpha = 1$. Then there are open intervals K_1 and K_2 with $\alpha \in \overline{K_1}$ and $\beta \in \overline{K_2}$ with either $T^2x \geqslant x$ for all $x \in K_1 \cup K_2$ or $T^2x \leqslant x$ for all $x \in K_1 \cup K_2$. Again, since $(T^2)'r = 0$, one can find open intervals H_1 and H_2 with $r \in \overline{H_1} \cap \overline{H_2}$ and with $T^2x < x$ if $x \in H_1$ and $T^2x > x$ if $x \in H_2$.

If $T^2x < x$ for $x \in K_1 \cup K_2$, then using H_1, one concludes that there is a fixed point of T^2 in (α, β) different from r_i, a contradiction. Analogously, if $T^2x \geqslant x$ for $x \in K_1 \cup K_2$, then using H_2, one arrives at another contradiction. Therefore $(T^2)'\alpha > 1$. □

PROPOSITION 3. *If $\{\alpha, \beta\}$ is the period-two cycle of (c_{i-1}, c_i), then $T^kx \to r_i$ as $k \to \infty$ for $x \in (\alpha, \beta)$.*

PROOF. If $x \in C_i$ and $C_i = [r_i, s_i]$, then T is monotone increasing and $Tx < x$ for $x \in (r_i, s_i\,]$; hence $TC_i \subset C_i$ and $x > Tx > T^2x > \cdots > r_i$. Consequently, $T^kx \to x^*$ which is a fixed point of T in C_i but the only such point is r_i. $T^kx \to r_i$. In the situation where $C_i = [s_i, r_i]$, $Tx > x$ for $x \in [\,s_i, r_i)$ and $x < Tx < T^2x < \cdots < r_i$. Again $T^kx \to r_i$. Suppose $x \in (\alpha, \beta) \setminus C_i$. If $T^kx \nrightarrow r_i$, then $\{T^kx\colon k \geqslant 0\} \cap C_i = \varnothing$.

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$\{T^k x\colon k \geqslant 0\} \subset (\alpha, \beta) \setminus C_i$ and one can assume that $x \in (\alpha, t_i)$ because the matrix V indicates that if $x \in (u_i, \beta)$, then $Tx \in (\alpha, t_i)$. From V, $\{T^{2j}x\colon j \geqslant 0\} \subset (\alpha, t_i)$ and $\{T^{2j+1}x\colon j \geqslant 0\} \subset (u_i, \beta)$. Because T is monotone decreasing on L_i and R_i, $T^2 y < T^2 z$ for $y < z$ and $y, z \in (\alpha, t_i)$ (or (u_i, β)). If $x < T^2 x$, then $Tx \geqslant T^3 x$ and $T^2 x < T^4 x$. Thus $x < T^2 x < T^4 x < \cdots < t_i$ and $Tx > T^3 x > T^5 x > \cdots > u_i$. $T^{2j}x \to x_1$ and $T^{2j+1}x \to x_2$. By continuity, $Tx_1 = x_2$ and $T^2 x_1 = x_1$. $\{x_1, Tx_1\}$ is a period-two cycle of (c_{i-1}, c_i) different from $\{\alpha, \beta\}$, a contradiction of Proposition 1. Therefore, $\{T^k x\colon k \geqslant 0\} \cap C_i \neq \varnothing$ and $T^k x \to r_i$. For $x = T^2 x$, one also concludes that $T^k x \to r_i$. If $x > T^2 x$, then $x > T^2 x > T^4 x > \cdots > \alpha$ and $Tx < T^3 x < T^5 x < \cdots < \beta$, and $T^{2j}x \to \alpha$ and $T^{2j+1}x \to \beta$. However,

$$\frac{T^{2(j+1)}x - T^2\alpha}{T^{2j}x - \alpha} = \frac{T^{2(j+1)}x - \alpha}{T^{2j}x - \alpha} < 1 \quad \text{and} \quad \lim_{j\to\infty} \frac{T^{2(j+1)}x - \alpha}{T^{2j}x - \alpha} \leqslant 1.$$

From Proposition 2, $(T^2)'\alpha > 1$ and one arrives at another contradiction. Therefore $x \leqslant T^2 x$, and $T^k x \to r_i$ for $x \in (\alpha, \beta)$. $\square$

With these three propositions, one can make a slight modification of $\mathcal{P}$ which will aid in proving Barna's Theorem. If α_i denotes one of the elements of the period-two cycle for (c_{i-1}, c_i), let $\mu = \min\{(T^2)'\alpha_i\colon i = 1, \ldots, n-2\}$. Since $\mu > 1$, choose λ so that $\mu > \lambda > 1$. Then there are $d_i \in (c_{i-1}, t_i)$ and $e_i \in (u_i, c_i)$ such that $(T^2)'d_i = (T^2)'e_i = \lambda$. If one now defines $L_i = (c_{i-1}, d_i)$, $R_i = (e_i, c_i)$ and $C_i = [d_i, e_i]$, the transition matrix A does not change, but one can deduce that $\mathcal{E}$, the exceptional set of points for T, is totally disconnected.

Let $x_0 x_1 x_2 \cdots x_i \cdots$ be a sequence determined by the matrix $V = [v_{ij}]$, i.e., $v_{x_i x_{i+1}} = 1$ for $i = 0, 1, 2, \ldots$ or $I_{x_i} \cap T^{-1} I_{x_{i+1}} \neq \varnothing$. As remarked before, $\bigcap_{i \geqslant 0} T^{-i} I_{x_i} \neq \varnothing$ for such a sequence. If $|I|$ denotes the length of I and

$$l = \max\{|L_i|, |R_i|\colon i = 1, \ldots, n-2\},$$

then

$$\left|I_{x_0} \cap T^{-1} I_{x_1} \cap T^{-2} I_{x_2} \cap \cdots \cap T^{2-j} I_{x_{2j}}\right| \leqslant \left|I_{x_0} \cap T^{-2} I_{x_2} \cap \cdots \cap T^{-2j} I_{x_{2j}}\right| \leqslant \lambda^{-j} l.$$

Thus $\bigcap_{i \geqslant 0} T^{-i} I_{x_i}$ consists of a single point, i.e., $\mathcal{E}$ is totally disconnected. (For convenience, $\mathcal{E}$ will be used for the exceptional set of points for T determined by V rather than $\mathcal{E} \setminus \mathcal{C}$.)

Because $\mathcal{P}$, the partition of $\mathbf{R}$, is dense in $\mathbf{R}$ and, for each $x \in \mathcal{E}$, there is a sequence of intervals $\{I_{x_i} \in \mathcal{P}\colon i = 0, 1, 2, \ldots\}$ for which $\{x\} = \bigcap_{i \geqslant 0} T^{-i} I_{x_i}$, $\mathcal{P}$ is a topological generator for $\mathcal{E}$ (see [**5**, p. 92]). Hence, for $X_0 \subset X$ defined by $X_0 = \{\mathbf{x} \in X\colon v_{x_i x_{i+1}} = 1 \text{ for all } i\}$, the set of $\mathbf{x}$ determined by V, the map $\pi\colon X_0 \to \mathbf{R}$ which assigns a sequence $\mathbf{x} \in X_0$ to a point of $\mathcal{E}$, is a homeomorphism by Theorem 5.12 in [**5**].

PROPOSITION 4. *$\mathcal{E}$ is a perfect set, i.e., a closed set for which each point is an accumulation point.*

PROOF. Because $\mathcal{E} \subset [r_0, r_{n-1}]$, $\mathcal{E}$ is compact. It suffices to prove that $\mathbf{x} \in X_0$ is an accumulation point using the metric $d(\mathbf{x}, \mathbf{y}) = \sum_{i \geqslant 0} 2^{-i} |x_i - y_i|$ for $\mathbf{x} = x_0 x_1 x_2 \cdots$, $\mathbf{y} = y_0 y_1 y_2 \cdots$ with $\mathbf{x}, \mathbf{y} \in X$. Let $\mathbf{x} = x_0 x_1 x_2 \cdots x_n \cdots \in X_0$. For $n \geqslant 1$, let

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$\mathbf{x}^n = x_0^n x_1^n x_2^n \cdots x_n^n x_{n+1}^n \cdots$ with $x_i^n = x_i$ for $i = 0, 1, \ldots, n$. Because V is irreducible, there is a finite sequence $x_n^n x_{n+1}^n \cdots x_{n+k}^n x_n^n$ where $v_{x_i^n x_{i+1}^n} = 1$ for $i = n, n + 1, \ldots, n + k - 1$ and $v_{x_{n+k}^n x_n^n} = 1$. By defining $\mathbf{x}^n = x_0^n x_1^n \cdots x_n^n x_{n+1}^n \cdots x_{n+k}^n x_n^n x_{n+1}^n \cdots x_{n+k}^n x_n^n$, $\mathbf{x}^n \in X_0$ and $(\mathbf{x}^n, \mathbf{x}) \leqslant 2^{-n} < 1/n$. Consequently, there is a sequence $\{\mathbf{x}^n\} \subset X_0$ such that $\mathbf{x}^n \to \mathbf{x}$. $\square$

Observations. (3) In the case where some of the roots also give rise to points of inflection, V remains the same; only the matrix W changes in that $L_i \cap T^{-1}C_i = \varnothing$ and $R_i \cap T^{-1}C_i = \varnothing$ if $r_i = s_i$.

(4) For the cases where $n = 1$ or 2, Newton's method always works except at the critical number in the case of $n = 2$, and the transition matrix points out the efficiency of the method.

(5) As for $n = 3$, the matrix V is reducible and indicates the existence of only a period-two cycle. From Proposition 1, there is only one such cycle. Another way of realizing that there can be only one is as follows: As pointed out by H. Shane, any cubic polynomial is symmetric with respect to its point of inflection; hence it suffices to consider $f(x) = x(x^2 - a^2)$ where $a > 0$. For such a polynomial, the period-two point must satisfy the condition $Tx = -x$. The solutions to this equation are $a/\sqrt{5}$ and $-a/\sqrt{5}$, and determine the cycle.

Extension of Barna's Theorem.

PROPOSITION 5. *Barna's Theorem is valid for* $f(x) = \prod_{i=0}^{k-1}(x_i + r_i)^{t_i}$ *where* $r_i \neq r_j$ $(i \neq j)$, $r_i \in \mathbf{R}$ *for* $i = 0, 1, \ldots, k - 1$, $t_i \geqslant 1$ *and* $k \geqslant 4$.

Observations. (6) Let r be a multiple root of f. Then $f(x) = (x - r)^t g(x)$ where $t \geqslant 2$ and $g(x) = \prod_{r_i \neq r}(x - r_i)^{t_i}$. By straightforward computation, one sees that r is a critical number of f but $T'r = (t - 1)/t > 0$. Thus, even though $f'(r) = 0$, $x = r$, is not a vertical asymptote of T, and the result of multiple roots is that the number of vertical asymptotes for T is exactly $(k - 1)$, say $c_0, c_2, \ldots, c_{k-2}$.

(7) If r_0 (or r_{k-1}) is a multiple root, then in (r_0, c_0) (or in (c_{k-2}, r_{k-1})), there is an s which gives rise to a point of inflection for f such that $T's = 0$. Consequently, T' is monotone increasing on $(-\infty, s)$ (or on (s, ∞)) and monotone decreasing on (s, c_0) (or on (c_{k-2}, s)) with $T'x \to -\infty$ as $x \to c_0^-$ (or as $x \to c_{k-2}^+$).

If r_i is a multiple root with $i \neq 0, k - 1$, then within (c_{i-1}, c_i) there are two values s_1^i and s_2^i where $f''(s_1^i) = f''(s_2^i) = 0$ and $T's_1^i = T's_2^i = 0$. Thus $T'x \to -\infty$ as $x \to c_{i=1}^+$ or as $x \to c_i^-$ and T is monotone increasing on (s_1^i, s_2^i) (see Figure 2).

(8) If one defines the partition $\mathcal{P}$ of $\mathbf{R}$ as before except that, for each (c_{i-1}, c_i) containing a multiple root, $L_i = (c_{i-1}, s_1^i)$, $R_i = (s_2^i, c_i)$ and $C_i = [s_1^i, s_2^i]$, then the transition matrix A is identical to the one that one derives for a polynomial of degree k having all real roots, each of multiplicity 1 (where k is the number of distinct roots of f).

In order to extend Barna's Theorem, it is necessary to prove

(1) for $a > 0$, $T'x = -a$

has exactly one solution in each L_i and each R_i. If $f(x) = \prod_{i=0}^{k-1}(x - r_i)^{t_i}$, then

$$f'(x) = f(x) \sum_{i=0}^{k-1} t_i/(x - r_i) = f(x) l(x)$$

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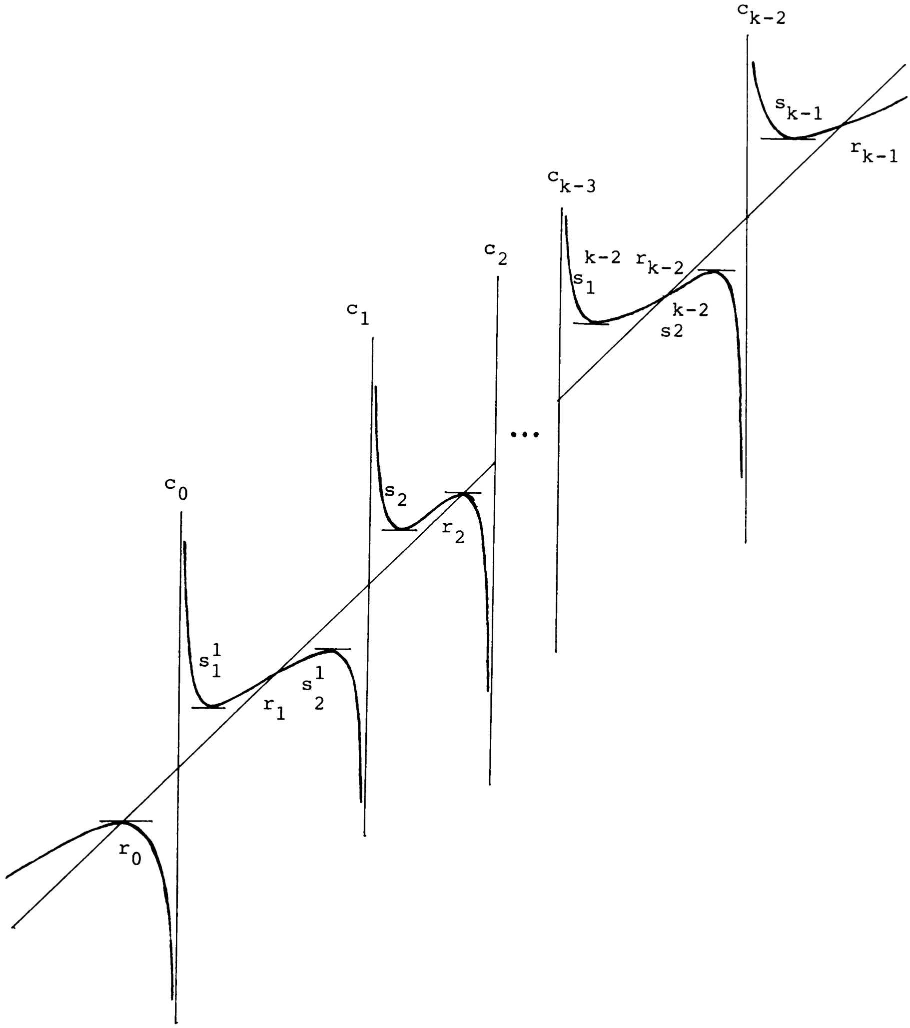

FIGURE 2. T with multiple roots at r_1, r_{k-2}, and r_{k-1}

and

$$f''(x) = f(x)\left[\left(\sum_{i=0}^{k-1} t_i/(x-r_i)\right)^2 - \sum_{i=0}^{k-1} t_i/(x-r_i)^2\right] = f(x)h(x).$$

Hence,

$$\begin{aligned} T'x &= f(x)f''(x)/[f'(x)]^2 = h(x)/[l(x)]^2 \\ &= h(x)\prod_{i=0}^{k-1}(x-r_i)^2 \Big/ \left\{[l(x)]^2 \prod_{i=0}^{k-1}(x-r_i)^2\right\} = N(x)D(x). \end{aligned}$$

One notices that $N(x)$ and $D(x)$ are polynomials of degree $2(k-1)$. Thus, (1) is equivalent to the equation $N(x) + aD(x) = 0$ which has at most $2(k-1)$ real solutions. Since $T'x \to -\infty$ as $x \to c_{i-1}^-$ and as $x \to c_i^+$ for $i = 1, \ldots, k-1$, $N(x) + aD(x) = 0$ has at least $2(k-1)$ real solutions. Therefore (1) is proven.

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With (1), all discussions involving the situation where all the roots are of multiplicity 1 remains true for the situation where all the roots are real not necessarily of multiplicity 1. Hence the extension of Barna's Theorem is true.

Barna's Theorem gives insight into the structure of $\mathfrak{S}$ for a Newton transform of a polynomial having all real roots. However, a bit more can be said about $\mathfrak{S}$ if the polynomial has at least 5 distinct roots. In particular, since $\mathfrak{S}$ contains all the periodic orbits of T, one can place a lower bound on the growth of the number of periodic orbits of T. To be precise, if $\mathrm{Fix}(T^n)$ denotes the number of fixed points of T^n, including those of T^m for $1 \leqslant m < n$, then $\mathrm{Fix}(T^n) > \exp(nb)$ for some number $b > 0$. The lower bound is found by using a theorem regarding the topological entropy of a continuous map of a circle into itself. (For a discussion of topological entropy see **[5]**.) In the present context, one can use the following interpretation of entropy denoted $h(T)$:

$$h(T) = \lim_{n \to \infty} (1/n) \log \mathrm{Fix}(T^n).$$

Since T: $\mathbf{R} \to \mathbf{R}$, one needs to transform T to a continous map of K, a circle of diameter 1, into K. One method is to apply the homeomorphism g: $K \to \mathbf{R}$ defined by

$$g(\theta) = \tan\left(\tfrac{1}{2} \cdot \theta\right) \in \mathbf{R} \quad \text{for } \theta \in [0, 2\pi)$$

(see **[6]**). As a continuous circle map, one realizes that the degree of T (see e.g. **[4]**) is $-(k-2)$ where k is the number of distinct real roots of f. If $k \geqslant 5$, then by **[3]**, $h(T) \geqslant \log|\deg T| = \log(k-2)$. (The result is for $k \geqslant 5$ because the case of $|\deg T| = 2$ is not resolved in **[3]**.) Since there is a lower bound for $h(T)$ with T: $K \to K$, one also has a lower bound for $h(T)$ when T: $\mathbf{R} \to \mathbf{R}$ because $h(T)$ is invariant under topological conjugacy.

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