DUNFORD-PETTIS OPERATORS
AND WEAK RADON-NIKODÝM SETS

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ABSTRACT. Let $K$ be a weak* -compact convex subset of a Banach space $X$. If every Dunford-Pettis operator from $L_1[0,1]$ into $X^*$ that maps the set \( \{ x_E/\mu(E) : E \text{ measurable}, \mu(E) > 0 \} \) into $K$ has a Pettis derivative, then $K$ is a weak Radon-Nikodym set. This positive answer to a question of M. Talagrand localizes a result of E. Saab.

In [6] Elias Saab showed that if every Dunford-Pettis operator from $L_1[0,1]$ into the dual of a Banach space $X$ has a Pettis derivative, then every operator from $L_1[0,1]$ into $X^*$ has a Pettis derivative so that consequently $X^*$ has the weak Radon-Nikodym property. The purpose of the present note is to answer a question of Talagrand by showing that this result localizes to weak* -compact convex subsets of arbitrary dual spaces.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. A function $f$ from $\Omega$ into a Banach space $X$ is said to be Pettis integrable if the scalar function $x^* f(\cdot)$ is integrable for each $x^*$ in the dual space $X^*$ and if for each measurable set $E$ in $\Sigma$ there is an element $x_E$ of $X$ that satisfies
\[
x^*(x_E) = \int_E x^* f \, d\mu.
\]
for every $x^*$ in $X^*$. In this case we write $x_E = \text{Pettis-}\int_E f \, d\mu$. A bounded subset $K$ of $X$ is called a weak Radon-Nikodym set [3] if for every finite measure space $(\Omega, \Sigma, \mu)$ and every bounded operator $S : L_1(\mu) \to X$ for which $S(x_E) \in \mu(E)K$ for every $E$ in $\Sigma$, there exists a Pettis integrable function $f : \Omega \to K$ such that $S(\psi) = \text{Pettis-}\int \psi f \, d\mu$ for every $\psi$ in $L_1(\mu)$. Such a function $f$ is called a Pettis derivative of the operator $S$. The Banach space $X$ has the weak Radon-Nikodym property if its closed unit ball, $B_X$, is a weak Radon-Nikodym set.

For the rest of this note we shall always take $K$ to be a weak* -compact convex subset of a dual space $X^*$.

THEOREM 1. Suppose $K$ is not a weak Radon-Nikodym set in $X^*$. Then there is a Dunford-Pettis operator $M : L_1[0,1] \to X^*$ such that $M(x_E) \in \mu(E)K$ for all Lebesgue measurable subsets $E$ of $[0,1]$ but such that $M$ does not have a Pettis derivative.

PROOF. We may assume without loss of generality that $0 \in K$. Since $K$ fails the weak Radon-Nikodym property, then so does the weak* -compact absolutely convex

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set $K - K$ [3]. Let $Y$ be the Banach space of continuous functions on $K - K$ and define a bounded linear operator $T: X \to Y$ by $Tx(x^*) = x^*(x)$ for every $x^*$ in $K - K$. Then $T^*(B_Y) = K - K$. Because $T^*(B_Y)$ is not a weak Radon-Nikodým set in $X^*$, the set $T(B_Y)$ is not weakly precompact in $Y$ [3]. (This may be thought of as a localization of the result that a Banach space does not contain a copy of $l_1$ if and only if its dual space has the weak Radon-Nikodým property.)

By Rosenthal's fundamental result [4] on weakly precompact sets, one can find a sequence $(Tx_n)$ in $T(B_X)$ that is equivalent to the usual $l_1$-basis $(e_n)$. Let $Y_0$ be the closed subspace of $Y$ spanned by the sequence $(Tx_n)$ and define $W: l_1 \to Y_0$ by $W(e_n) = Tx_n$. Then $W$ defines an isomorphism from $l_1$ onto $Y_0$, hence its adjoint $W^*: Y_0^* \to l_\infty$ has a bounded inverse. Define $R: l_1 \to X$ by $R(e_n) = x_n$ and let $i: Y_0 \to Y$ denote the natural inclusion map. A moment’s reflection reveals that $R^*T^* = W^*i^*$.

Now $Y_0^*$ fails the weak Radon-Nikodým property since $l_1$ embeds in $Y_0$. By Saab’s global result, then, there exists a Dunford-Pettis operator from $L_1[0,1]$ into $Y_0^*$ that is not Pettis representable. The operator that Saab constructed in [6] has the form $QP\ N$ where $N, P$ and $Q$ are defined on the spaces illustrated in the following diagram (and the other operators are to be specified shortly).

The operator $N$ is a Dunford-Pettis operator from $L_1[0,1]$ into $L_1[0,1]$ but $QP\ N: L_1[0,1] \to Y_0^*$ has no Pettis derivative. We may assume that all three operators have norm 1. Because $M[0,1] = C[0,1]^*$ is an $L$-space, the operator $Q$ has a norm-preserving lifting $S$ from $M[0,1]$ to $Y^*$ such that $i^*S = Q$ [2, Theorem 2].

Since $T^*SP(x_E/\mu(E))$ belongs to $T^*(B_Y)$, a splitting argument due to E. Saab [5] and based on a compactness argument of Lindenstrauss produces two operators $U, V: L_1[0,1] \to X^*$ such that $U(x_E/\mu(E))$ and $V(x_E/\mu(E))$ belong to $K$ and such that $T^*SP = U - V$. Bourgain has shown [1] that the positive part of a Dunford-Pettis operator from $L_1[0,1]$ into $L_1[0,1]$ is also a Dunford-Pettis operator. We may therefore assume that $N$ is a positive operator, for if both $QP\ N^+$ and $QP\ N^-$ are Pettis representable then so is $QP\ N$. Thus

$$N(x_E/\mu(E)) \subset \{ f \in L_1[0,1]: f \geq 0, \| f \| \leq 1 \} = \mathcal{P}.$$  

Because 0 belongs to $K$ and because $U(x_E/\mu(E)) \subset K$ for all nonnull measurable sets $E$, a convexity argument shows that $U(\mathcal{P}) \subset K$. Therefore we have $UN(x_E/\mu(E)) \subset K$ and similarly $VN(x_E/\mu(E)) \subset K$ for all nonnull measurable sets. Also notice that both $UN$ and $VN$ are Dunford-Pettis operators.

Now suppose both $UN$ and $VN$ have a Pettis derivative. Then so does $T^*SPN =$...
UN – VN and also \( R^*T^*SPN \). But \( R^*T^*SPN = W^*i^*SPN = W^*QPN \) and \( W^* \) has a bounded inverse, implying that \( QPN \) would have a Pettis derivative. This contradiction shows that either \( M = UN \) or \( M = VN \) must fail to have a Pettis derivative and finishes the proof.

The set \( K \) has the weak Radon-Nikodým property if every operator taking the normalized characteristic functions into \( K \) has a Pettis derivative. This is also equivalent to requiring that all operators taking the normalized characteristic functions into \( K \) be Dunford-Pettis operators [5]. We now see that Theorem 1 connects these two characterizations in the following sense.

**Theorem 2.** A weak*-compact convex subset \( K \) of a dual space \( X^* \) is a weak Radon-Nikodým set if and only if every Dunford-Pettis operator from \( L_{1}[0,1] \) into \( X^* \) that maps the normalized characteristic functions into \( K \) has a Pettis derivative.

**References**


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