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DUNFORD-PETTIS OPERATORS AND WEAK RADON-NIKODÝM SETS

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ABSTRACT. Let K be a weak*-compact convex subset of a Banach space X. If every Dunford-Pettis operator from $L_1[0,1]$ into X^* that maps the set $\{\chi_E/\mu(E): E \text{ measurable}, \mu(E) > 0\}$ into K has a Pettis derivative, then K is a weak Radon-Nikodým set. This positive answer to a question of M. Talagrand localizes a result of E. Saab.

In [**6**] Elias Saab showed that if every Dunford-Pettis operator from $L_1[0,1]$ into the dual of a Banach space X has a Pettis derivative, then every operator from $L_1[0,1]$ into X^* has a Pettis derivative so that consequently X^* has the weak Radon-Nikodým property. The purpose of the present note is to answer a question of Talagrand by showing that this result localizes to weak*-compact convex subsets of arbitrary dual spaces.

Let (Ω, Σ, μ) be a finite measure space. A function f from Ω into a Banach space X is said to be Pettis integrable if the scalar function $x^*f(\cdot)$ is integrable for each x^* in the dual space X^* and if for each measurable set E in Σ there is an element x_E of X that satisfies

$$x^*(x_E) = \int_E x^* f \, d\mu$$

for every x^* in X^*. In this case we write $x_E = \text{Pettis-}\int_E f\,d\mu$. A bounded subset K of X is called a weak Radon-Nikodým set [**3**] if for every finite measure space (Ω, Σ, μ) and every bounded operator $S: L_1(\mu) \to X$ for which $S(\chi_E) \in \mu(E)K$ for every E in Σ, there exists a Pettis integrable function $f: \Omega \to K$ such that $S(\psi) = \text{Pettis-}\int \psi f\,d\mu$ for every ψ in $L_1(\mu)$. Such a function f is called a Pettis derivative of the operator S. The Banach space X has the weak Radon-Nikodým property if its closed unit ball, B_X, is a weak Radon-Nikodým set.

For the rest of this note we shall always take K to be a weak*-compact convex subset of a dual space X^*.

THEOREM 1. *Suppose K is not a weak Radon-Nikodým set in X^*. Then there is a Dunford-Pettis operator $M: L_1[0,1] \to X$ such that $M(\chi_E) \in \mu(E)K$ for all Lebesgue measurable subsets E of $[0,1]$ but such that M does not have a Pettis derivative.*

PROOF. We may assume without loss of generality that $0 \in K$. Since K fails the weak Radon-Nikodým property, then so does the weak*-compact absolutely convex

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set $K - K$ [**3**]. Let Y be the Banach space of continuous functions on $K - K$ and define a bounded linear operator $T: X \to Y$ by $Tx(x^*) = x^*(x)$ for every x^* in $K - K$. Then $T^*(B_{Y^*}) = K - K$. Because $T^*(B_{Y^*})$ is not a weak Radon-Nikodým set in X^*, the set $T(B_X)$ is not weakly precompact in Y [**3**]. (This may be thought of as a localization of the result that a Banach space does not contain a copy of l_1 if and only if its dual space has the weak Radon-Nikodým property.)

By Rosenthal's fundamental result [**4**] on weakly precompact sets, one can find a sequence (Tx_n) in $T(B_X)$ that is equivalent to the usual l_1-basis (e_n). Let Y_0 be the closed subspace of Y spanned by the sequence (Tx_n) and define W: $l_1 \to Y_0$ by $W(e_n) = Tx_n$. Then W defines an isomorphism from l_1 onto Y_0, hence its adjoint W^*: $Y_0^* \to l_\infty$ has a bounded inverse. Define R: $l_1 \to X$ by $R(e_n) = x_n$ and let i: $Y_0 \to Y$ denote the natural inclusion map. A moment's reflection reveals that $R^*T^* = W^*i^*$.

Now Y_0^* fails the weak Radon-Nikodým property since l_1 embeds in Y_0. By Saab's global result, then, there exists a Dunford-Pettis operator from $L_1[0,1]$ into Y_0^* that is not Pettis representable. The operator that Saab constructed in [**6**] has the form QPN where N, P and Q are defined on the spaces illustrated in the following diagram (and the other operators are to be specified shortly).

$$\begin{tikzcd}
L_1[0,1] \arrow[r, "N"] & L_1[0,1] \arrow[r, "P"] \arrow[drrr, bend right, "{U,V}"'] & M[0,1] \arrow[r, "Q"] \arrow[dr, "S"'] & Y_0^* \arrow[r, "W^*"] & \ell_\infty \\
 & & & Y^* \arrow[u, "i^*"'] \arrow[r, "T^*"'] & X^* \arrow[u, "R^*"']
\end{tikzcd}$$

The operator N is a Dunford-Pettis operator from $L_1[0,1]$ into $L_1[0,1]$ but QPN: $L_1[0,1] \to Y_0^*$ has no Pettis derivative. We may assume that all three operators have norm 1. Because $M[0,1] = C[0,1]^*$ is an L-space, the operator Q has a norm-preserving lifting S from $M[0,1]$ to Y^* such that $i^*S = Q$ [**2**, Theorem 2].

Since $T^*SP(\chi_E/\mu(E))$ belongs to $T^*(B_{Y^*}) = K - K$, a splitting argument due to E. Saab [**5**] and based on a compactness argument of Lindenstrauss produces two operators U, V: $L_1[0,1] \to X^*$ such that $U(\chi_E/\mu(E))$ and $V(\chi_E/\mu(E))$ belong to K and such that $T^*SP = U - V$. Bourgain has shown [**1**] that the positive part of a Dunford-Pettis operator from $L_1[0,1]$ into $L_1[0,1]$ is also a Dunford-Pettis operator. We may therefore assume that N is a positive operator, for if both QPN^+ and QPN^- are Pettis representable then so is QPN. Thus

$$N(\chi_E/\mu(E)) \subset \{f \in L_1[0,1]: f \geq 0, \|f\| \leq 1\} = \mathcal{P}.$$

Because 0 belongs to K and because $U(\chi_E/\mu(E)) \in K$ for all nonnull measurable sets E, a convexity argument shows that $U(\mathcal{P}) \subset K$. Therefore we have $UN(\chi_E/\mu(E)) \in K$ and similarly $VN(\chi_E/\mu(E)) \in K$ for all nonnull measurable sets. Also notice that both UN and VN are Dunford-Pettis operators.

Now suppose both UN and VN have a Pettis derivative. Then so does $T^*SPN =$

$UN - VN$ and also R^*T^*SPN. But $R^*T^*SPN = W^*i^*SPN = W^*QPN$ and W^* has a bounded inverse, implying that QPN would have a Pettis derivative. This contradiction shows that either $M = UN$ or $M = VN$ must fail to have a Pettis derivative and finishes the proof.

The set K has the weak Radon-Nikodým property if every operator taking the normalized characteristic functions into K has a Pettis derivative. This is also equivalent to requiring that all operators taking the normalized characteristic functions into K be Dunford-Pettis operators [**5**]. We now see that Theorem 1 connects these two characterizations in the following sense.

THEOREM 2. *A weak***-compact convex subset K of a dual space X^* is a weak Radon-Nikodým set if and only if every Dunford-Pettis operator from $L_1[0,1]$ into X^* that maps the normalized characteristic functions into K has a Pettis derivative.*

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