PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 2, June 1984

IMAGE AREAS AND H_2 NORMS OF ANALYTIC FUNCTIONS

SHŌJI KOBAYASHI

ABSTRACT. For an analytic function f in the unit disc U with $f(0) = 0$, the inequality $\|f\|_2^2 \leq \frac{1}{\pi}$ area $\{f(U)\}$ is shown, where an equality occurs if and only if f is a constant multiple of an inner function. As a corollary, it is shown that for an analytic function in a general domain the square of its H_2 norm is bounded by its Dirichlet integral, with the equality condition being settled

1. Introduction. Let U denote the unit disc and T the unit circle in the complex plane $\mathbf{C}$. It is shown by Alexander, Taylor and Ullman [**1**, Theorem 1, p. 335] that if f is an analytic function in U with $f(0) = 0$, then

$$\|f\|_2^2 \leq \frac{1}{\pi}\text{area}\{f(U)\} \tag{1.1}$$

holds, where $\|f\|_2$ denotes the H_2 norm

$$\|f\|_2 = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right\}^{1/2} \tag{1.2}$$

of f. An *inner function* is a bounded analytic function ψ in U such that the *Fatou's boundary function* ψ^* satisfies $|\psi^*(e^{i\theta})| = 1$ almost everywhere on T. In this paper we shall offer another proof of (1.1) and show that an equality occurs in (1.1) if and only if f is of a form $f = c\psi$, where c is a constant and ψ is an inner function with $\psi(0) = 0$. Our proof depends on *Littlewood's subordination principle* and *Green's formula*.

Let D be a plane domain with $0 \in D$, which possesses a Green's function. Following Rudin [**5**], we denote by $H_2(D)$ the class of functions F analytic in D for which $|F(z)|^2$ admits a *harmonic majorant* in D. For $F \in H_2(D)$, let $h(z)$ be the *least harmonic majorant* of $|F(z)|^2$ in D and define

$$\|F\|_2 = h(0)^{1/2}. \tag{1.3}$$

It is well known that (1.3) reduces to (1.2) when $D = U$.

In §2 we state our main results. In §3 we state preliminary lemmas, which we use in §4 for the proof of the Main Theorem. In §5, as a corollary to the Main Theorem, the inequality

$$\|F\|_2^2 \leq \frac{1}{\pi} \int\int_D |F'(z)|^2 \, dx \, dy$$

is shown for $F \in H_2(D)$ with $F(0) = 0$, and its equality condition is settled.

Received by the editors July 7, 1983.
1980 *Mathematics Subject Classification*. Primary 30D55, 30C80.

2. Main results. Now we state our main results in the form of a theorem.

MAIN THEOREM. *If $f \in H_2(U)$ with $f(0) = 0$, then*

$$\|f\|_2^2 \leq \frac{1}{\pi}\operatorname{area}\{f(U)\},$$

and an equality occurs if and only if f is of the form $f = c\psi$, where c is a constant and ψ is an inner function with $\psi(0) = 0$.

COROLLARY. *If $F \in H_2(D)$ with $F(0) = 0$, then*

$$\|F\|_2^2 \leq \frac{1}{\pi}\int\int_D |F'(z)|^2\,dx\,dy, \tag{2.1}$$

where an equality occurs if and only if D is a domain which is obtained from a simply-connected domain W by deleting a set of capacity zero and F is (extended to) a conformal map of W onto a disc with center at 0.

3. Preliminary results. In this section we state several preliminary lemmas which we use in the next section for the proof of the Main Theorem.

LEMMA 1. *If ϕ is a bounded analytic function in U with $|\phi(z)| \leq 1$, $z \in U$, and $\phi(0) = 0$, then for any $f \in H_2(U)$*

$$\|f \circ \phi\|_2 \leq \|f\|_2, \tag{3.1}$$

where an equality occurs for some nonconstant $f \in H_2(U)$ if and only if ϕ is an inner function.

PROOF. The inequality (3.1) is an easy consequence of Littlewood's subordination principle (see, say, [**2**, p. 10]). For the equality condition, see [**6**, Theorem 3, p. 351 or **4**, Theorem 1, p. 316].

Let p: $U \to D$ be a *universal covering map* of D such that $p(0) = 0$. The following lemma is well known, for the proof see [**5**, p. 50 or **4**, Lemma 1, p. 316].

LEMMA 2. *If $F \in H_2(D)$ with $F(0) = 0$, then*

$$\|F\|_2 = \|F \circ p\|_2. \tag{3.2}$$

The next lemma is an essential part of our proof of the Main Theorem.

LEMMA 3. *If $I(z) \equiv z$, $z \in D$, then*

$$\|I\|_2^2 \leq \frac{1}{\pi}\operatorname{area}(D), \tag{3.3}$$

and an equality occurs if and only if D is a domain of a form $D = \{|z| < r\} - E$, where $r > 0$ and E is a closed set of capacity zero.

PROOF. Let S be a plane domain with smooth boundary Γ, and u and v be C^2 functions on the closure $\overline{S}$ of S. Then Green's theorem states that

$$\int\int_S (v\Delta u - u\Delta v)\,dx\,dy = \int_\Gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\,ds, \tag{3.4}$$

where Δ denotes the Laplacian, $\partial/\partial n$ differentiation in the inner normal direction and ds arc length on Γ.

Let $h(z)$ denote the least harmonic majorant of $|I(z)|^2 \equiv |z|^2$ in D. Let $G(z)$ be a Green's function of D with logarithmic singularity at 0, and $G^*(z)$ be its harmonic conjugate, which is locally defined up to an additive constant. Put $P(z) = G(z) + iG^*(z)$, then it is known that $P(z)$ is multiple-valued but it has a single-valued derivative $P'(z)$, which is analytic in D, except for a simple pole at 0.

First we assume that D is a *regular domain*, where "*regular*" means that the boundary B of D consists of a finite number of mutually disjoint analytic Jordan curves. Let k be a positive integer. We apply Green's formula (3.4) in D with $u(z) = |z|^2$ and $v(z) = 1 - e^{-2kG(z)}$. Simple calculations show that $\Delta u = 4$ and $\Delta v = -4k^2|P'(z)|^2 e^{-2kG(z)}$ in D and that $v = 0$ and $\partial v/\partial n = 2k\partial G(z)/\partial n$ on B, which, substituted into (3.4), yields

$$\begin{aligned} &\int\int_D (1 - e^{-2kG(z)})\,dx\,dy + k^2 \int\int_D |z|^2|P'(z)|^2 e^{-2kG(z)}\,dx\,dy \\ &\quad = \frac{k}{2}\int_B |z|^2 \frac{\partial G(z)}{\partial n}\,ds = \pi k h(0). \end{aligned} \tag{3.5}$$

Put $g(z) = ze^{P(z)}$, then it is multiple-valued but $|g(z)|^2$ is single-valued and subharmonic in D, since $\log|g(z)|^2 = 2\log|z| + 2G(z)$ is harmonic. We again apply Green's formula (3.4) in D with $u(z) = |g(z)|^2$ and $v(z) = 1 - e^{-2(k+1)G(z)}$. Similarly simple calculations show that $\Delta u = 4|g'(z)|^2$ and $\Delta v = -4(k+1)^2|P'(z)|^2 e^{-2(k+1)G(z)}$ in D and that $v = 0$ and $\partial v/\partial n = 2(k+1)\partial G(z)/\partial n$ on B, which, substituted into (3.4), yields

(3.6)

$$\begin{aligned} &\int\int_G |g'(z)|^2(1 - e^{-2(k+1)G(z)})\,dx\,dy + (k+1)^2 \int\int_D |z|^2\,|P'(z)|^2 e^{-2kG(z)}\,dx\,dy \\ &\quad = \frac{k+1}{2}\int_B |z|^2 \frac{\partial G(z)}{\partial n}\,ds = \pi(k+1)h(0), \end{aligned}$$

since $|g(z)|^2 = |z|^2 e^{2G(z)}$ for $z \in D$ and $|g(z)|^2 = |z|^2$ for $z \in B$. Here note that the two integrals of the second terms of (3.5) and (3.6) are identical. Therefore, combining the two equalities, we see

$$\begin{aligned} &\int\int_D (1 - e^{-2kG(z)})\,dx\,dy \\ &\quad = \frac{\pi k}{k+1}h(0) + \left(\frac{k}{k+1}\right)^2 \int\int_D |g'(z)|^2(1 - e^{-2(k+1)G(z)})\,dx\,dy. \end{aligned}$$

On letting $k \to \infty$, Lebesgue's monotone convergence theorem yields

$$\frac{1}{\pi}\text{area}(D) = h(0) + \frac{1}{\pi}\int\int_D |g'(z)|^2\,dx\,dy. \tag{3.7}$$

In order to deal with the case of a general domain D, let $\{D_n\}$ be a regular exhaustion of D such that $0 \in D_n$, $n = 1, 2, \ldots$. We denote by $G_n(z)$ and $g_n(z)$, respectively, the functions for D_n which correspond to $G(z)$ and $g(z)$ for D. Let $h_n(z)$ be the least harmonic majorant of $|I(z)|^2 \equiv |z|^2$ in D_n. Then (3.7) for D_n is

$$\frac{1}{\pi}\text{area}(D_n) = h_n(0) + \frac{1}{\pi}\int\int_{D_n} |g_n'(z)|^2\,dx\,dy,$$

in which, letting $n \to \infty$, we see by Lebesgue's monotone convergence theorem and Fatou's lemma

$$\frac{1}{\pi}\text{area}(D) \geq h(0) + \frac{1}{\pi}\int\int_D |g'(z)|^2\, dx\, dy, \tag{3.8}$$

since $h_n(z) \to h(z)$ and $|g_n'(z)| \to |g'(z)|$ for any $z \in D$ as $n \to \infty$. Noting (1.3), we see that (3.8) implies (3.3).

As for the equality condition, the if part is almost trivial. In fact, if D is as stated in the lemma, then we easily see that $h(z) \equiv r^2$, and hence

$$\|I\|_2^2 = r^2. \tag{3.9}$$

On the other hand, $\text{area}(D) = \pi r^2$, since a set of capacity zero is of area zero. Thus, an equality occurs in (3.3). Next, assume that an equality holds in (3.3), then we see by (3.8) that $|g'(z)| \equiv 0$ in D, and hence $G(z) = \log(c/|z|)$ for some positive constant c, which means that D is as stated in the lemma. This completes the proof of the lemma.

4. Proof of Main Theorem. Let $D = f(U)$ and $p\colon U \to D$ be a universal covering map of D such that $p(0) = 0$, as before. By the monodromy theorem, we can determine a single-valued branch of $p^{-1} \circ f$, which we denote by ϕ. Then we easily see that ϕ is a bounded analytic function in U with $|\phi(z)| \leq 1,\ z \in U$, for which $f = p \circ \phi$, and hence by Lemma 1

$$\|f\|_2 \leq \|p\|_2. \tag{4.1}$$

Applying Lemma 2 with $f = I$, we see

$$\|I\|_2 = \|I \circ p\|_2 = \|p\|_2, \tag{4.2}$$

which, combined with (4.1) and Lemma 3, yields (1.1), as asserted.

As for the equality condition, the if part is again almost trivial. In fact, suppose that f is of the form as stated in the theorem. Since any inner function covers U with the exception of a set of capacity zero by a theorem of Frostman [**3**], we easily see

$$\frac{1}{\pi}\ \text{area}\{f(U)\} = |c|^2 = \|f\|_2^2.$$

Next assume that an equality holds in (1.1); then equalities must hold both in (4.1) and (3.3). Therefore, by the equality conditions of Lemmas 1 and 3, we see that ϕ must be an inner function and that $f(U)$ must be a domain of the form as stated in Lemma 3. Thus, we see that $|p(z)| \leq r$ for $z \in U$ and that $\|p\|_2 = \|T\|_2 = r$ by (4.2) and (3.9), and hence that p must be expressed as $p = r\psi$, where ψ is an inner function. Since the composite function of two inner functions is again an inner function by Lemma 2 (cf. [**6**, p. 351]), we see that f must be the form as stated in the theorem.

5. Proof of Corollary. Put $f = F \circ p$; then by Lemma 2

$$\|f\|_2 = \|F\|_2. \tag{5.1}$$

It is obvious that

$$\text{area}\{F(D)\} \leq \int\int_D |F'(z)|^2\, dx\, dy \tag{5.2}$$

and that an equality occurs in (5.2) if and only if F is univalent in D. Since $F(D) = f(U)$, we obtain (2.1) by combining (5.1), (5.2) and the theorem. The equality condition immediately follows those of (5.2) and the theorem.

REMARK. With a slight modification of the above argument, we can also prove a version of the corollary for the case of Riemann surfaces.

REFERENCES

1. H. Alexander, B. A. Taylor and J. L. Ullman, *Areas of projections of analytic sets,* Invent. Math. **16** (1972), 335–341.
2. P. L. Duren, *Theory of H_p spaces,* Academic Press, New York, 1970.
3. O. Frostman, *Potentiel d'équilibre et capacité des ensembles avec quelques applications á la théorie des fonctions,* Medd. Lunds Univ. Mat. Sem. **3** (1935), 1–118.
4. S. Kobayashi and N. Suita, *On subordination of subharmonic functions,* Kodai Math. J. **3** (1980), 315–320.
5. W. Rudin, *Analytic functions of class H_p,* Trans. Amer. Math. Soc. **78** (1955), 46–66.
6. J. V. Ryff, *Subordinate H_p functions,* Duke Math. J. **33** (1966), 347–354.

DEPARTMENT OF ELECTRIC ENGINEERING, TECHNOLOGICAL UNIVERSITY OF NAGAOKA, NAGAOKA, JAPAN