ABSTRACT. D. S. Ornstein has shown that if the ergodic maximal function of a nonnegative function is in $L^1$ then the function is in $L \log^+ L$. This paper gives a new simple proof of this fact.

In [1] a simple proof of the maximal ergodic theorem was presented. This paper uses the method of [1] to give a new proof of D. S. Ornstein’s $L \log^+ L$ result.

NOTATION. In a probability space $(X, \Sigma, m)$, let $T: X \to X$ be a measurable ($A \in \Sigma \Rightarrow T^{-1}A \in \Sigma$) measure preserving ($m(A) = m(T^{-1}A)$) ergodic ($TA = A \Rightarrow A = \emptyset$ or $A = X$) transformation.

The ergodic maximal function, $f^*(x)$ is defined on a probability space $(X, \Sigma, m)$, for nonnegative $f$, by

$$f^*(x) = \sup_{n>0} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

In this setting it has been shown that

THEOREM (ORNSTEIN [2]). If the ergodic maximal function $f^*$ is integrable, then the function $f$ is in the class $L \log^+ L$.

PROOF. The proof follows, as in Ornstein’s paper [2], from the following result.

THEOREM (ORNSTEIN [2]). Let $f \geq 0$ and assume $f \in L^1(X)$. Then

$$m\{f^* > \lambda\} \geq \frac{1}{2\lambda} \int_{\{f^* > \lambda\}} f(x) \, dm(x).$$

PROOF. Using the Dominated Convergence Theorem and a truncation argument, we can assume $f \in L^\infty(X)$. Inequality (1) can be rewritten as

$$\int_{\{f^* > \lambda\}} f(x) - 2\lambda \, dm(x) \leq 0.$$

Let $g(x) = f(x) - \lambda$ and the problem becomes to show

$$\int_{\{g^* > 0\}} g(x) - \lambda \, dm(x) \leq 0.$$
Define $E = \{g^* > 0\}$, and study

$$\int_E g(x) - \lambda \, dm(x) = \int_X (g(x) - \lambda) \chi_E(x) \, dm(x)$$

$$= \frac{1}{L} \sum_{k=0}^{L-1} \int (g(T^k x) - \lambda) \chi_E(T^k x) \, dm(x)$$

$$= \int \frac{1}{L} \sum_{k=0}^{L-1} (g(T^k x) - \lambda) \chi_E(T^k x) \, dm(x).$$

We now must consider the sum which occurs in (2). To study this we introduce the stopping time, $\tau(x)$, defined by $\tau(x) = \inf_{n>0} \{n| T^n x \notin E\}$.

**Lemma.** If $x \notin E$ then

$$\sum_{k=0}^{L-1} (g(T^k x) - \lambda) \chi_E(T^k x) \leq 0. \quad (3)$$

**Proof.** Let $x \notin E$ and $\tau(x) > 1$. Assume $\tau(x) < L$. (If not replace $\tau(x)$ by $L$ in the following argument.) Then we have

$$\tau(x) - 1 \sum_{k=0}^{\tau(x)-1} g(T^k x) \leq 0$$

but since $g(x) > -\lambda$ we also have

$$-\lambda + \sum_{k=1}^{\tau(x)-1} g(T^k x) \leq 0$$

and hence certainly

$$\sum_{k=1}^{\tau(x)-1} (g(T^k x) - \lambda) \leq 0. \quad (4)$$

Now we can estimate (3) by decomposing the sum into "blocks" where $\chi_E(T^k x) = 1$ and blocks where $\chi_E(T^k x) = 0$. Define $r(x) = \inf\{n| T^n x \in E\}$ to be the time of first return to $E$. Then

$$\sum_{k=0}^{L-1} (g(T^k x) - \lambda) \chi_E(T^k x) = \sum_{k=0}^{\tau(x)-1} (g(T^k x) - \lambda)(0) + \sum_{k=r(x)}^{\tau(r(x))-1} (g(T^k x) - \lambda)(1)$$

$$+ \sum_{k=r(r(x))}^{\tau(r(r(x)))-1} (g(T^k x) - \lambda)(0) + \sum_{k=r(r(r(x)))}^{\tau(r(r(r(x))))-1} (g(T^k x) - \lambda)(1)$$

$$+ \cdots,$$

where the last term has an upper index of $L-1$. Each of the nonzero sums above is a case of (4), i.e. if the bottom index is $b$, replace $x$ in (4) by $T^b x$. Thus each term is either nonpositive or zero. \(\square\)
We are now ready to estimate (2). For $x \notin E$, by the Lemma, the sum is nonpositive. If $x \in E$ then write

$$\frac{1}{L} \sum_{k=0}^{L-1} (g(T^k x) - \lambda) \chi_E(T^k x) = \frac{1}{L} \sum_{k=0}^{(\tau(x)-1)\wedge L-1} (g(T^k x) - \lambda) \chi_E(T^k x)$$

Then argument in the Lemma shows the second term on the right is nonpositive. Consequently only the first term is a problem. Define $B_n = \{x \in E \mid \tau(x) > n\}$. Since $T$ is ergodic we know $m(B_n) \to 0$. Given $\varepsilon > 0$ select $n$ so large that $m(B_n) < \varepsilon/(2\|g - \lambda\|_{\infty})$. Next, select $L$ so large that $n\|g - \lambda\|_{\infty}/L < \varepsilon/2$. The required estimate for (2) follows by (1) integrating over $B_n$ and using the $L^\infty$ estimate for $g - \lambda$; (2) integrating over $E - B_n$ and using the fact that the estimate for the first term on the right-hand side of (5) is less than $\varepsilon/2$ and the last term is nonpositive; (3) integrating over $E^c$, where by the Lemma the function is nonpositive; and (4) letting $\varepsilon \to 0$.

References


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