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ORNSTEIN'S $L\log^+ L$ THEOREM

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ABSTRACT. D. S. Ornstein has shown that if the ergodic maximal function of a nonnegative function is in L^1 then the function is in $L\log^+ L$. This paper gives a new simple proof of this fact.

In [**1**] a simple proof of the maximal ergodic theorem was presented. This paper uses the method of [**1**] to give a new proof of D. S. Ornstein's $L\log^+ L$ result.

NOTATION. In a probability space (X, Σ, m), let $T: X \to X$ be a measurable $(A \in \Sigma \Rightarrow T^{-1}A \in \Sigma)$ measure preserving $(m(A) = m(T^{-1}A))$ ergodic $(TA = A \Rightarrow A = \emptyset$ or $A = X)$ transformation.

The ergodic maximal function, $f^*(x)$ is defined on a probability space (X, Σ, m), for nonnegative f, by

$$f^*(x) = \sup_{n>0} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

In this setting it has been shown that

THEOREM (ORNSTEIN [**2**]). *If the ergodic maximal function f^* is integrable, then the function f is in the class $L\log^+ L$.*

PROOF. The proof follows, as in Ornstein's paper [**2**], from the following result.

THEOREM (ORNSTEIN [**2**]). *Let $f \geq 0$ and assume $f \in L^1(X)$. Then*

$$m\{f^* > \lambda\} \geq \frac{1}{2\lambda} \int_{\{f^*>\lambda\}} f(x)\, dm(x). \tag{1}$$

PROOF. Using the Dominated Convergence Theorem and a truncation argument, we can assume $f \in L^\infty(X)$. Inequality (1) can be rewritten as

$$\int_{\{f^*>\lambda\}} f(x) - 2\lambda\, dm(x) \leq 0.$$

Let $g(x) = f(x) - \lambda$ and the problem becomes to show

$$\int_{\{g^*>0\}} g(x) - \lambda\, dm(x) \leq 0.$$

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Define $E = \{g^* > 0\}$, and study

$$\int_E g(x) - \lambda\, dm(x) = \int_X (g(x) - \lambda)\chi_E(x)\, dm(x)$$

$$= \frac{1}{L}\sum_{k=0}^{L-1}\int (g(T^k x) - \lambda)\chi_E(T^k x)\, dm(x) \tag{2}$$

$$= \int \frac{1}{L}\sum_{k=0}^{L-1}(g(T^k x) - \lambda)\chi_E(T^k x)\, dm(x).$$

We now must consider the sum which occurs in (2). To study this we introduce the stopping time, $\tau(x)$, defined by $\tau(x) = \inf_{n>0}\{n \mid T^n x \notin E\}$.

LEMMA. *If $x \notin E$ then*

$$\sum_{k=0}^{L-1}(g(T^k x) - \lambda)\chi_E(T^k x) \le 0. \tag{3}$$

PROOF. Let $x \notin E$ and $\tau(x) > 1$. Assume $\tau(x) < L$. (If not replace $\tau(x)$ by L in the following argument.) Then we have

$$\sum_{k=0}^{\tau(x)-1} g(T^k x) \le 0$$

but since $g(x) > -\lambda$ we also have

$$-\lambda + \sum_{k=1}^{\tau(x)-1} g(T^k x) \le 0$$

and hence certainly

$$\sum_{k=1}^{\tau(x)-1} (g(T^k x) - \lambda) \le 0. \tag{4}$$

Now we can estimate (3) by decomposing the sum into "blocks" where $\chi_E(T^k x) = 1$ and blocks where $\chi_E(T^k x) = 0$. Define $r(x) = \inf\{n \mid T^n x \in E\}$ to be the time of first return to E. Then

$$\sum_{k=0}^{L-1}(g(T^k x) - \lambda)\chi_E(T^k x) = \sum_{k=0}^{r(x)-1} (g(T^k x) - \lambda)(0) + \sum_{k=r(x)}^{\tau(r(x))-1} (g(T^k x) - \lambda)(1)$$
$$+ \sum_{k=\tau(r(x))}^{r(\tau(r(x)))-1} (g(T^k x) - \lambda)(0) + \sum_{k=r(\tau(r(x)))}^{\tau(r(\tau(r(x))))-1} (g(T^k x) - \lambda)(1)$$
$$+ \cdots,$$

where the last term has an upper index of $L - 1$. Each of the nonzero sums above is a case of (4), i.e. if the bottom index is b, replace x in (4) by $T^b x$. Thus each term is either nonpositive or zero. □

We are now ready to estimate (2). For $x \notin E$, by the Lemma, the sum is nonpositive. If $x \in E$ then write

$$\frac{1}{L}\sum_{k=0}^{L-1}(g(T^k x)-\lambda)\chi_E(T^k x) = \frac{1}{L}\sum_{k=0}^{(\tau(x)-1)\wedge L-1}(g(T^k x)-\lambda)\chi_E(T^k x) + \frac{1}{L}\sum_{k=\tau(x)}^{L-1}(g(T^k x)-\lambda)\chi_E(T^k x). \tag{5}$$

Then argument in the Lemma shows the second term on the right is nonpositive. Consequently only the first term is a problem. Define $B_n = \{x \in E \mid \tau(x) > n\}$. Since T is ergodic we know $m(B_n) \to 0$. Given $\varepsilon > 0$ select n so large that $m(B_n) < \varepsilon/(2\|g-\lambda\|_\infty)$. Next, select L so large that $n\|g-\lambda\|_\infty/L < \varepsilon/2$. The required estimate for (2) follows by (1) integrating over B_n and using the L^∞ estimate for $g-\lambda$; (2) integrating over $E - B_n$ and using the fact that the estimate for the first term on the right-hand side of (5) is less than $\varepsilon/2$ and the last term is nonpositive; (3) integrating over E^c, where by the Lemma the function is nonpositive; and (4) letting $\varepsilon \to 0$.

References

1. R. L. Jones, *New proofs for the maximal ergodic theorem and the Hardy-Littlewood maximal theorem,* Proc. Amer. Math. Soc. **87** (1983), 681–684.
2. D. S. Ornstein, *A remark on the Birkhoff ergodic theorem,* Illinois J. Math. **15** (1971), 77–79.

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