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A BOUNDARY VALUE PROBLEM FOR SYMMETRIC ELLIPTIC SYSTEMS OF FIRST ORDER SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS IN OPEN SETS OF CLASS C^1

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ABSTRACT. Si dimostrano un teorema di unicità ed un teorema di esistenza per sistemi ellittici simmetrici a coefficienti costanti di equazioni alle derivate parziali del primo ordine semilineari in aperti di classe C^1 e con dati al contorno in L^p.

Introduction. An existence and uniqueness theorem was established in [**3**] for a boundary value problem related to an elliptic system of first order semilinear partial differential equations. The problem was considered in a bounded open subset Ω of $\mathbf{R}^m$ $(m \geq 3)$ of class C^2 and boundary data of class C^1.

The study of a boundary value problem for first order linear elliptic systems with constant coefficients in bounded open sets of class C^1 carried out by the present authors in [**7**], suggested the extension of the results obtained in [**3**] to the case in which Ω is of class C^1 and the boundary data are in L^p.

In this paper, we prove an existence and uniqueness theorem for that problem, making use of the integral representation used by A. Avantaggiati in [**2**] for the solutions of the linear system considered there.

1. Consider the following problem:

I. Let Ω be a bounded open subset of $\mathbf{R}^m$ $(m \geq 3)$ of class C^1; let $f = (f_r)_{1\leq r\leq 2n}$ be a finite sequence of functions such that

(i) $f_r(X,u)$ is continuous in u for almost every $X \in \Omega$ and is measurable in X for any $u \in \mathbf{R}^{2n}$;

(ii) if $2 \leq s < +\infty$ and $t = sm/(m-1)$, $ab > 0$, an $h \in]1-\frac{1}{t}, 1[$ and an $a_r \in L^s(\Omega)$ exist such that

$$|f_r(X,u)| \leq a_r(x) + b|u|^h; \tag{1.1}$$

let $B = (b_{kq})_{1\leq k\leq n; 1\leq q\leq 2n}$ be a matrix whose elements are of class $C^0(\partial\Omega)$ with rank equal to n at any point of $\partial\Omega$ and let $b_0 = (b_{k0})_{1\leq k\leq n} \in (L^s(\partial\Omega))^n$. Our problem is to determine a $2n$-tuple $u = (u_q)_{1\leq q\leq 2n} \in (L^t(\Omega))^{2n}$ with first order derivatives a.e. in Ω, which satisfies an elliptic symmetric system of first order semilinear differential equations with real constant coefficients

$$\sum_{q=1}^{2n}\sum_{p=1}^{m} a_{rq}^p \frac{\partial u_q}{\partial x_p}(X) = f_r(X,u), \qquad r = 1,\ldots,2n,$$

a.e. in Ω, and is such that

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(i_1) for any $\alpha \in]0,1[$, a $\delta > 0$ exists such that for any $q \in \{1,\ldots,2n\}$ the maximal nontangential function of u_q,

$$u_q^*(P) = \sup\{|u_q(X)| : X \in C_\alpha(P) \cap B(P,\delta)\},$$

belongs to $L^s(\partial\Omega)$ (here $C_\alpha(P) = \{X \in \Omega : (X-P)\cdot N(P) > \alpha|X-P|\}$ where $N(P)$ is the inner normal at P to $\partial\Omega$ and $B(P,\delta)$ is the open ball centered at P and with radius δ),

(i_2) for almost every $P \in \partial\Omega$ and for any $q \in \{1,\ldots,2n\}$, the limit

$$\lim_{X\to P;X\in C_\alpha(P)} u_q(X)$$

exists; if we denote this limit by $u_q(P)$ it satisfies the following conditions a.e. in Ω:

$$\sum_{q=1}^{2n} b_{kq}(P)u_q(P) = b_{k0}(P), \qquad k = 1,\ldots,n. \tag{1.3}$$

2. Let

$$a(N(P)) = (a_{rq}(N(P)))_{1\le r,q\le 2n} = \left(\sum_{p=1}^{m} a_{rq}^p N_p(P)\right)_{1\le r,q\le 2n} \tag{2.1}$$

where $N(P) = (N_1(P),\ldots,N_m(P))$. Denote by B^* the transpose of B, by I the unit matrix of order $2n$ and by 0 the null matrix of order n. If

$$D(P,\rho) = \det\begin{pmatrix} a(N(P)) - \rho I & B^*(P) \\ B(P) & 0 \end{pmatrix} \tag{2.2}$$

the following theorem holds.

THEOREM 2.1. *If for any $P \in \partial\Omega$ the equation $D(P,\rho) = 0$ admits only positive (resp. negative) roots and if $u \in \mathbf{R}^{2n} \mapsto f(X,u)$ is an increasing (resp. decreasing) monotone function for almost every $X \in \Omega$,*[2] *then problem* I *has at most one solution.*

PROOF. If u^1 and u^2 are solutions of problem I, since we have, a.e. in Ω,

$$\sum_{r,q=1}^{2n}\sum_{p=1}^{m} a_{rq}^p \frac{\partial}{\partial x_p}[(u_q^1(X) - u_q^2(X))\cdot(u_r^1(X) - u_r^2(X))]$$
$$= 2(f(X,u^1(X)) - f(X,u^2(X)))\cdot(u^1(X) - u^2(X))$$

and since, due to (i) and (ii), $f(\cdot,u^i(\cdot)) \in L^{t^1}(\Omega)$ $(i = 1,2)$ with $1/t^1 + 1/t = 1$, we have that the function

$$\sum_{r,q=1}^{2n}\sum_{p=1}^{m} a_{rq}^p (u_q^1(X) - u_q^2(X))\cdot(u_r^1(X) - u_r^2(X))$$

belongs to $W^{1,1}(\Omega)$. Hence, by means of arguments similar to those used in [**3**, Theorem 2.1], this theorem follows.

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[2]If H is a Hilbert space, we say that $f\colon H \to H$ is an increasing (resp. decreasing) monotone function, if it satisfies the following condition: $\forall u,v \in H\colon u \ne v \Rightarrow (f(u) - f(v))\cdot(u-v) > 0$ (resp. < 0).

3. Let $M=(M_{rq})_{1\leq r,q\leq 2n}$ be the fundamental matrix of the system (1.2) defined by (5.1″) in [**2**]. We note that the following properties were proved in [**2**]:

(α_1) $M_{rs}(\lambda X)=\lambda^{1-m}M_{rs}(X)$, $\lambda>0$,

(α_2) $M_{rs}(-X)=-M_{rs}(X)$,

(α_3) M_{rs} *is an analytic function in* $\mathbf{R}^m-\{0\}$.

Let f be the vector which appears in (1.2). The following proposition holds:

PROPOSITION 3.1. *If we set*

$$F_1e(X)=\int_\Omega M(X-Y)f(Y,e(Y))\,dY, \tag{3.1}$$

$$F_2\psi(X)=\int_{\partial\Omega} M(Q-X)\psi(Q)\,dQ, \tag{3.2}$$

a.e. in Ω, *we have that*

(i_3) F_1 *and* F_2 *are continuous operators from* $(L^t(\Omega))^{2n}$ *into* $(H^{1,s}(\Omega))^{2n}$ *and from* $(L^s(\partial\Omega))^{2n}$ *into* $(L^t(\Omega))^{2n}$,[3] *respectively,*

(i_4) *a constant* $c>0$ *exists such that*

$$\|F_1e\|_{(H^{1,s}(\Omega))^{2n}}\leq c(\|a\|_{(L^s(\Omega))^{2n}}+\|e\|^h_{(L^t(\Omega))^{2n}}), \tag{3.3}$$

$$\|F_2\psi\|_{(L^t(\Omega))^{2n}}\leq c\|\psi\|_{(L^s(\partial\Omega))^{2n}} \tag{3.4}$$

with $a=(a_r)_{1\leq r\leq 2n}$.

PROOF. The statement for F_2 is an obvious consequence of Proposition 2.v in [**4**] which holds also in the case in which Ω is of class C^1. Furthermore, by (i) and (ii) the map $e\mapsto f(\cdot,e(\cdot))$ is continuous from $(L^t(\Omega))^{2n}$ into $(L^s(\Omega))^{2n}$ and by 3.IV in [**4**] the operator $\alpha\mapsto\int_\Omega M_{rq}(X-Y)\alpha(Y)dY$ is continuous from $L^s(\Omega)$ into $H^{1,s}(\Omega)$. These facts immediately imply inequality (3.3).

In the sequel we will assume that for the quadratic form

$$\sum_{r,q=1}^{2n} a_{rq}(N(P))u_ru_q, \tag{3.5}$$

the following hypothesis is satisfied:

(I_1) Any root of the equation $\det(a(N(P))-\rho I)=0$ has a constant multiplicity with respect to $P\in\partial\Omega$.

Hence a finite covering $\{B_1,\dots,B_N\}$ of $\partial\Omega$ exists made up of coordinate neighborhoods and for any $j\in\{1,\dots,N\}$ a matrix $(d^j_{qr})_{1\leq q\leq 2n,1\leq r\leq n}=d^j$ of functions of class $C^0(B_j)$ exists with rank n at any point of B_j and for any $B_j\cap B_i\neq\emptyset$ an orthogonal matrix $(\vartheta^{ji}_{lk})_{1\leq l,k\in n}$ of class $C^0(B_j\cap B_i)$ exists such that

$$d^i_{qr}=\sum_{l=1}^n\vartheta^{ji}_{lr}\cdot d^j_{ql},\qquad \forall r\in\{1,\dots,n\}\,\forall q\in\{1,\dots,2n\}$$

(see n.5 of [**7**]).

[3] $H^{1,s}(\Omega)$ is the completion of $C^1(\overline{\Omega})$ with respect to the norm

$$\|f\|_{H^{1,s}(\Omega)}=\|f\|_{L^s(\Omega)}+\sum_{|\alpha|=1}\|D^\alpha f\|_{L^s(\Omega)}.$$

We set, for any $e \in (L^t(\Omega))^{2n}$ and any $\varphi \in L^s(\{\vartheta\}_B)$[4,5]

$$(3.6)\quad u(X) = \int_{\partial\Omega} M(Q-X)(d\cdot\varphi)(Q)\,dQ + \int_{\Omega} M(X-Y) f(Y, e(Y))\,dY \quad \text{a.e. in } \Omega.$$

If we require the vector u to satisfy the boundary conditions (1.3) and if we take into account (3.5) in [**6**] and $(\alpha_1), (\alpha_2), (\alpha_3)$, for almost every $P \in \partial\Omega$, we have

$$(3.7)\qquad B\cdot C\cdot (d\cdot\varphi)(P) + \int_{\partial\Omega}^{*} B(P) M(P-Q)(d\cdot\varphi)(Q)\,dQ = g(P)$$

where

$$(3.8)\quad \int_{\partial\Omega}^{*} B(P)M(P-Q)(d\cdot\varphi)(Q)dQ = \lim_{\epsilon\to 0^+} \int_{\partial\Omega - I(P,\epsilon)} B(P)\cdot M(P-Q)(d\cdot\varphi)(Q)dQ,$$

$$(3.9)\quad C(P) = -\int_{\pi_p}^{*} M(P-Q+N(P))dQ = -\lim_{\epsilon\to\infty}\int_{\pi_p - C(P,\epsilon)} M(P-Q+N(P))dQ,$$

$$(3.10)\qquad g(P) = b_0(P) - B(P)\int_{\Omega} M(P-Y) f(Y, e(Y))\,dY,$$

$I(P,\epsilon)$ is the portion of $\partial\Omega$ having as a projection on the tangent plane π_p to $\partial\Omega$ at P, the ball with center P and radius $\epsilon > 0$ and $C(P,\epsilon)$ is the ball of π_p with center P and radius $\epsilon > 0$.

REMARK 3.1. From Proposition 3.1 and trace theorems it follows that the function defined in (3.10) belongs to $(L^s(\partial\Omega))^n$ and

$$(3.11)\qquad \|g\|_{(L^s(\partial\Omega))^n} \le c_1(\|b_0\|_{(L^s(\partial\Omega))^n} + \|a\|_{(L^s(\Omega))^{2n}} + \|e\|^h_{(L^t(\Omega))^{2n}}).$$

In the sequel we assume the symbolic matrices

$$\mathcal{M}^j(P,\tau) = B(P)\cdot\Psi(P,\tau)\cdot d^j(P) \qquad \forall P\in B_j\ \forall\tau\in\mathbf{R}^{m-1} \text{ with } |\tau| = 1$$

associated with the singular integral equations (3.7), where $\Psi(P,\tau)$ is the symbolic matrix of the system of singular integral operators on $\partial\Omega$ (see n.3 of [**7**]) given by[6]

[4] $L^s(\{\vartheta\}_B) = \{\varphi = (\varphi^j)_{1\le j\le N} \in \prod_{j=1}^N (L^j(B_j))^n \colon \varphi^j = \vartheta^{ji}\varphi^i$ a.e. in $B_i \cap B_j\}$.

[5] $d\cdot\varphi$ is the vector of $(L^s(\partial\Omega))^{2n}$ whose restriction to B_j coincides with $d^j\cdot\varphi^j$.

[6] The symbol of the following operator $\mathcal{A}_{ns}$ on $\partial\Omega$

$$\mathcal{A}_{rs} f(P) = f(P)\int_{\pi_p}^{*} M_{rs}(P-Q+N(P))\,dQ + \int_{\partial\Omega}^{*} M_{rs}(P-Q) f(Q)\,dQ$$

is the function on the cotangent bundle of $\partial\Omega$ defined by

$$\Psi_{rs}(P, \textstyle\sum \xi_i\,dx_i) = a_{rs}(x(P)) + \lim_{\epsilon\to 0^+}\int_{\epsilon<|\eta|<1/\epsilon} e^{i\xi\cdot\eta} h(x(P),\eta)\,d\eta$$

where

$$a_{rs}(x) = \sqrt{1+|\nabla\varphi(x)|^2}\Big(\int_{|z|<1} M_{rs}(z, 1+\nabla\varphi(x)\cdot z)\,dz + \int_{|z|>1}[M_{rs}(z,1+\nabla\varphi(x)\cdot z) - M_{rs}(z,\nabla\varphi(x)\cdot z)]\,dz\Big),$$

$$h(x,z) = \sqrt{1+|\nabla\varphi(x)|^2}\, M_{rs}(z,\nabla\varphi(x)\cdot z)$$

and P is in a coordinate neighborhood V with coordinates x such that with respect to this coordinate system

$$V\cap\Omega = \{(x,t) : x\in\mathbf{R}^{m-1}, t > \varphi(x)\}\cap V$$

where $\varphi \in C_0^1(\mathbf{R}^{m-1})$, $\varphi(0) = \partial\varphi(0)/\partial x_i = 0$, $i = 1,\dots,m-1$.

$$\mathcal{A}\colon f \in (L^s(\partial\Omega))^{2n} \to C(P)\cdot f(P) + \int_{\partial\Omega}^{*} M(Q-P)f(Q)\,dQ,$$

satisfy the following assumption

(I_2) For any $P \in \partial\Omega$, $\tau \in \mathbf{R}^{m-1}$ with $|\tau| = 1$ and $P \in B_j$, $\mathcal{M}^j(P,\tau) \neq 0$.

Furthermore, suppose that for any $P \in \partial\Omega$ the equation $D(P,\rho) = 0$ admits only positive (resp. negative) roots and that this further assumption is satisfied:

(I_3) For almost every point $P \in \partial\Omega$ the quadratic form

$$\sum_{j,l=1}^{n}\sum_{r,q=1}^{2n} A_{rq}(N(P))b_{jr}(P)b_{lq}(P)\lambda_j\lambda_l$$

is positive (resp. negative) definite (where $(A_{rq}(N(P)))_{1\le r,q\le 2n}$ is the inverse matrix of $a(N(P))$).

Under the above assumptions the function S which to any $\varphi \in L^s(\{\vartheta\}_B)$ associates the element of $(L^s(\partial\Omega))^n$ which appears on the left-hand side of (3.7) is injective and the equation $S(\varphi) = g$ with g given by (3.10) admits one and only one solution thanks to Remark (5.3) in [**7**]. Furthermore S is surjective: actually, by an argument similar to that used in [**2**] and taking into account the results obtained in [**7**], we obtain that the theorem holds for $s = 2$, since the transposed homogeneous system associated with $S\varphi = g$ has only the trivial solution. For $s \geq 2$, if $g \in (L^s(\partial\Omega))^n$ and $\varphi \in L^2(\{\vartheta\}_B)$ is the only solution of the equation $S\varphi = g$, by an argument similar to that used to show Theorem 5.1 in [**7**] it follows that $\varphi \in L^s(\{\vartheta\}_B)$. These facts imply that S^{-1} is continuous and then an $H > 0$ exists such that

$$\|\varphi\|_{L^s(\{\vartheta\}_B)} \le H\|g\|_{(L^s(\partial\Omega))^n}. \tag{3.12}$$

PROPOSITION 3.2. *If we set*

$$T(e) = F_1(e) + F_2(d\cdot\varphi) \tag{3.13}$$

where φ is the unique solution of system (3.7), *then*

(i_5) *T is a continuous operator from $(L^t(\Omega))^{2n}$ into itself,*

(i_6) *a $c_2 > 0$ exists such that*

$$\|T(e)\|_{(L^t(\Omega))^{2n}} \le c_2(\|b_B\|_{(L^s(\partial\Omega))^n} + \|a\|_{(L^s(\Omega))^{2n}} + \|e\|^h_{(L^t(\Omega))^{2n}}). \tag{3.14}$$

PROOF. Let $(e_l)_{l\in\mathbf{N}}$ be a sequence of elements of $(L^t(\Omega))^{2n}$ converging there to e. From Proposition 3.1 it follows that $(F_1(e_l))_{l\in\mathbf{N}}$ converges to $F_1(e)$ in $(H^{1,s}(\Omega))^{2n}$. As a consequence, $(F_1(e_l))_{l\in\mathbf{N}}$ converges to $F_1(e)$ in $(L^t(\Omega))^{2n}$ and

$$\int_\Omega M(P-Y)f(Y,e_l(Y))\,dY$$

converges to $\int_\Omega M(P-Y)f(Y,e(Y))\,dY$ in $(L^s(\partial\Omega))^{2n}$. The statement ($i_5$) follows from (3.10) and the continuity of S^{-1} and F_2, while (i_6) is an obvious consequence of (3.3), (3.4), (3.11) and (3.12).

Choose $\rho > 0$ such that[7]

$$\|e\|_{(L^t(\Omega))^{2n}} \le \rho \Rightarrow \|T(e)\|_{(L^t(\Omega))^{2n}} \le \rho. \tag{3.15}$$

[7]The existence of ρ is ensured by (3.13) since $h \in]0,1[$.

PROPOSITION 3.3. *With the same notations as in Proposition* 3.2, *if we set*

$$E = \{e \in (L^t(\Omega))^{2n} : ||e||_{(L^t(\Omega))^{2n}} \le \rho\},$$

then

(i_7) $T(E) \subset E$,

(i_8) *The restriction of* T *to* E *has at least one fixed point.*

PROOF. (i_7) is an immediate consequence of (3.14). As far as (i_8) is concerned, since E is closed and convex, it suffices, thanks to the Schauder fixed point theorem, to show that $T(E)$ is relatively compact. To this aim let $(e_l)_{l\in\mathbf{N}}$ be a sequence of elements of E. Since, by (3.3), $(F_1(e_l))_{l\in\mathbf{N}}$ is bounded in $(H^{1,s}(\Omega))^{2n}$, it follows that $(F_1(e_l))_{l\in\mathbf{N}}$ has a subsequence which converges in $(L^t(\Omega))^{2n}$ thanks to the Sobolev embedding theorems and also $(\int_\Omega M(P-Y)f(Y,e_l(Y))\,dY)_{l\in\mathbf{N}}$ has a subsequence which converges in $(L^s(\partial\Omega))^{2n}$ thanks to the well-known trace theorems. From (3.10), (3.13) and the continuity of S^{-1} and F_2 it follows that $(T(e_l))_{l\in\mathbf{N}}$ has a subsequence converging in $(L^t(\Omega))^{2n}$.

At last the following theorem holds:

THEOREM 3.1. *Problem* I *admits at least one solution.*

PROOF. If u is a fixed point of T (see Proposition 3.3), then u is a solution of Problem I.

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