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AN INTEGRAL MEAN ESTIMATE FOR POLYNOMIALS

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ABSTRACT. An inequality involving

$$\max_{|z|=1} |P'(z)| \quad \text{and} \quad \left(\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right)^{1/q}, \qquad q > 0,$$

for a polynomial $P(z)$ having all its zeros in $|z| \leq 1$ is presented.

1. Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. Concerning the estimate of $|P'(z)|$ when there is a restriction on the zeros of $P(z)$, Turan [**9**] proved the following:

THEOREM A. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then*

$$\frac{n}{2} \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |P'(z)|. \tag{1}$$

The result is best possible and equality in (1) *holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.*

In this paper we present a generalization of Theorem A in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact, our result is related to the following one of Saff and Sheil-Small [**8**]:

THEOREM B. *If $P(z)$ is a polynomial of degree n having all its zeros on $|z| = 1$, then for each $q > 0$*

$$\left(\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right)^{1/q} \leq (A_q)^{1/q} \left(\max_{|z|=1} |P(z)|/2\right) \tag{2}$$

where

$$A_q = 2^{q+1} \sqrt{\pi} \Gamma\left(\tfrac{1}{2}q + \tfrac{1}{2}\right) / \Gamma\left(\tfrac{1}{2}q + 1\right).$$

The result is best possible and equality in (2) *holds if and only if $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.*

In the proof of Theorem B, Saff and Sheil-Small established the inequality

$$n \left(\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right)^{1/q} \leq (A_q)^{1/q} \max_{|z|=1} |P'(z)| \tag{3}$$

for the polynomials $P(z)$ having all its zeros on $|z| = 1$, and then in (3) used the Erdös-Lax theorem [**3**] to arrive at (2).

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2. We now present our main result:

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for each $q > 0$*

$$n\left(\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right)^{1/q} \leq (A_q)^{1/q} \max_{|z|=1} |P'(z)| \tag{4}$$

where $A_q = 2^{q+1}\sqrt{\pi}\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)/\Gamma\left(\frac{1}{2}q + 1\right)$. The result is best possible and equality in (4) holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta|$.

REMARK 1. For a polynomial $P(z)$ having all its zeros in $|z| > 1$, an inequality like (4) is not expected because the right-hand side of (4) may become arbitrarily small while $|P(z)|$ is uniformly close to 1 on $|z| = 1$. This follows from a result due to MacLane [**4**]: for any $\varepsilon > 0$ there exists a sequence of polynomials $T_k(z)$ having all its zeros on $|z| = 1 + \varepsilon$ and $T_k(z) \to 1$ uniformly on $|z| \leq 1 + \varepsilon'$ as $k \to \infty$, $\varepsilon' < \varepsilon$. This observation makes Theorem 1 interesting.

REMARK 2. Making $q \to \infty$ in (4), one gets (1).

PROOF OF THEOREM 1. Let $P(z)$ be a polynomial of degree n. Define $Q(z) = z^n\overline{P(1/\overline{z})}$. It is immediate that $|P(z)| = |Q(z)|$ for $|z| = 1$. Moreover, $P(z) = z^n\overline{Q(1/\overline{z})}$ so

$$P'(z) = nz^{n-1}\overline{Q(1/\overline{z})} - z^{n-2}\overline{Q'(1/\overline{z})},$$

from which

$$z^{n-1}\overline{P'(1/\overline{z})} = nQ(z) - zQ'(z).$$

Since $|z^{n-1}\overline{P'(1/\overline{z})} = |P'(z)|$ for $|z| = 1$, it further implies that

$$|P'(z)| = |nQ(z) - zQ'(z)|. \tag{5}$$

We also have

$$n|Q(z)| = |nQ(z) - zQ'(z) + zQ'(z)| \tag{6}$$

$$= |nQ(z) - zQ'(z)|\,|1 + w(z)| \tag{7}$$

where $w(z) = zQ'(z)/(nQ(z) - zQ'(z))$. Hence for $|z| = 1$,

$$n|P(z)| = |P'(z)|\,|1 + w(z)|. \tag{8}$$

Next, we show that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then $w(z)$ is analytic in $|z| \leq 1$ and $1 + w(z)$ is subordinate to $1 + z$ for $|z| \leq 1$.

Since $P(z)$ has all its zeros in $|z| \leq 1$, by the Gauss-Lucas theorem $P'(z)$ also has all its zeros in $|z| \leq 1$. This implies $z^{n-1}\overline{P'(1/\overline{z})} = nQ(z) - zQ'(z) \neq 0$ in $|z| < 1$. Moreover, if $|\alpha| = 1$ is a zero of order k $(< n)$ of $nQ(z) - zQ'(z)$, then from (5) it is also a zero of order k of $P'(z)$. As the unit disk which contains all the zeros of $P(z)$ is strictly convex, following the Gauss-Lucas theorem, α is a zero of order $k+1$ of $P(z)$ and so of $Q(z)$. Hence α is a zero of order k of $Q'(z)$. This concludes that $w(z)$ is analytic in $|z| \leq 1$. To show that $|w(z)| \leq 1$ for $|z| = 1$ we note [**5**] that $|Q'(z)| \leq |P'(z)|$ for $|z| = 1$. Also $w(0) = 0$. Thus $1 + w(z)$ is subordinate to $1 + z$ for $|z| \leq 1$.

Applying a well-known property of subordination [**2**] to (8) we deduce that

$$n\left(\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right)^{1/q} \le \max_{|z|=1} |P'(z)| \left(\int_0^{2\pi} |1+w(e^{i\theta})|^q \, d\theta\right)^{1/q}$$
$$\le \max_{|z|=1} |P'(z)| (A_q)^{1/q}.$$

This completes the proof of Theorem 1.

3. By using the techniques in the above proof, we prove

THEOREM 2. *If $P(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}(|P'(z)| + |Q'(z)|) = n \max_{|z|=1} |P(z)| \tag{9}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Moreover, the maximums of both sides in (9) are attained at the same point $|z_0| = 1$.

PROOF OF THEOREM 2. Let $P(z)$ be a polynomial of degree n. It is known [**5**] that for $|z| \le 1$

$$|P'(z)| + |Q'(z)| \le n \max_{|z|=1} |P(z)|. \tag{10}$$

We also note that from (5) and (6) one gets

$$\begin{aligned} n|P(z)| &= |zP'(z) + nP(z) - zP'(z)| \\ &\le |P'(z)| + |nP(z) - zP'(z)| = |P'(z)| + |Q'(z)| \end{aligned} \tag{11}$$

for $|z| = 1$. (10) and (11) imply (9). In view of (11), it is obvious that if $|P(z_0)| = \max_{|z|=1} |P(z)|$ and $|z_0| = 1$, then $|P'(z_0)| + |Q'(z_0)| = n|P(z_0)|$. This proves Theorem 2.

If $P(z)$ is a self-inverse polynomial of degree n, i.e. $P(z) = z^n\overline{P(1/\overline{z})}$, from Theorem 2 we get an already known result:

THEOREM C. *If $P(z)$ is a self-inverse polynomial of degree n, then*

$$\max_{|z|=1} |P'(z)| = \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{12}$$

Theorem C is proved by Govil [**1**], O'Hara and Rodrigues [**6**], and Saff and Sheil-Small [**8**] independent of each other. In this case, we further note that $|P'(z_0)| = (n/2)|P(z_0)|$, where $|z_0| = 1$ and $|P(z_0)| = \max_{|z|=1} |P(z)|$. This implies that if $|P(z)|$ attains its maximum on $|z| = 1$ at n points (n is the degree of $P(z)$) then $|P'(z)|$ must also attain its maximum at those n points. Then $P'(z)$ being a polynomial of degree $n-1$ must reduce to $P'(z) = \alpha z^{n-1}$ where α is a constant. Consequently, we have

THEOREM 3. *If $P(z)$ is a self-inverse polynomial of degree n and $|P(z)|$ attains its maximum at n points on $|z| = 1$, then $P(z) = \alpha z^n + \overline{\alpha}$ where α is a constant.*

REMARK 3. Rubinstein [**7**] has recently proved Theorem 3 for the class of polynomials having all its zeros on $|z| = 1$.

Finally, we give an application of Theorem C to real trigonometric polynomials. If $t(\theta)$ is a real trigonometric polynomial of order n, then

$$e^{in\theta}t(\theta) = R(e^{i\theta}) \tag{13}$$

where $R(z)$ is a self-inverse polynomial of degree $2n$. Differentiating (13) with respect to θ and applying Theorem C, we get

$$\max_{\theta} \sqrt{\{nt(\theta)\}^2 + \{t'(\theta)\}^2} = n \max_{\theta} |t(\theta)|,$$

an inequality essentially due to van der Corput and Schaake [**10**].

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