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A FUNDAMENTAL INEQUALITY IN THE CONVOLUTION OF L_2 FUNCTIONS ON THE HALF LINE

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ABSTRACT. For any positive integer q and $F_j \in L_2(0,\infty)$, we note the inequality for the iterated convolution $\prod_{j=1}^{2q} * F_j$ of F_j:

$$\int_0^\infty \left| \prod_{j=1}^{2q} * F_j(t) \right|^2 t^{1-2q}\, dt \leq \frac{1}{(2q-1)!} \prod_{j=1}^{2q} \int_0^\infty |F_j(t)|^2\, dt.$$

1. Result. For the convolution $F * G$ of $F \in L_p(-\infty,\infty)$ and $G \in L_1(-\infty,\infty)$ $(p \geq 1)$, we know the fundamental inequality

$$\|F * G\|_p \leq \|F\|_p \|G\|_1. \tag{1.1}$$

See, for example, [**4**, p. 3]. Note that for $F, G \in L_2(-\infty,\infty)$, in general, $F * G \notin L_2(-\infty,\infty)$. In this note, we give

THEOREM 1.1. *We take a positive integer q and $F_j \in L_2(0,\infty)$. Then, for the iterated convolution $\prod_{j=1}^{2q} * F_j$ of F_j, we obtain the inequality*

$$\int_0^\infty \left| \prod_{j=1}^{2q} * F_j(t) \right|^2 t^{1-2q}\, dt \leq \frac{1}{(2q-1)!} \prod_{j=1}^{2q} \int_0^\infty |F_j(t)|^2\, dt. \tag{1.2}$$

Equality holds here if and only if each F_j is expressible in the form $c_j e^{-t\bar{u}}$ for some constant c_j and for some point u such that $\operatorname{Re} u > 0$ and u is independent of j.

2. Proof of theorem. For $F \in L_2(0,\infty)$ and, in general, $q \geq {}_2$, we consider the integral transform

$$f(z) = \int_0^\infty e^{-zt} F(t) t^{q-(1/2)}\, dt. \tag{2.1}$$

Then it is known that $f(z)$ are analytic on $\operatorname{Re} z > 0$, the images $f(z)$ form a Hilbert space H_q admitting the reproducing kernel

$$K_q(z,\overline{u}) = \Gamma(2q)/(z+\overline{u})^{2q} \tag{2.2}$$

and

$$\|f\|_{H_q}^2 = \int_0^\infty |F(t)|^2\, dt. \tag{2.3}$$

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See, for example, [**2**, pp. 113–114]. In particular, when $q = \frac{1}{2}$, we set

$$f_j(z) = \int_0^\infty e^{-zt} F_j(t)\, dt, \tag{2.4}$$

and, therefore

$$\|f_j\|^2_{H_{1/2}} = \int_0^\infty |F_j(t)|^2\, dt. \tag{2.5}$$

Note that for an integer q,

$$K_q(z, \overline{u}) = (2q-1)! K_{1/2}(z, \overline{u})^{2q}. \tag{2.6}$$

Hence, from the general theory of Aronszajn [**1**, pp. 357–362, 350–352], we obtain the inequality

$$\left\| \prod_{j=1}^{2q} f_j \right\|^2_{H_q} \leq \frac{1}{(2q-1)!} \prod_{j=1}^{2q} (\|f_j\|^2_{H_{1/2}}) \tag{2.7}$$

as in [**3**].

On the other hand, we have

$$\prod_{j=1}^{2q} f_j(z) = \int_0^\infty e^{-zt} \left(\prod_{j=1}^{2q} * F_j(t) \right) dt, \tag{2.8}$$

and since $\prod_{j=1}^{2q} f_j(z) \in H_q$,

$$\prod_{j=1}^{2q} f_j(z) = \int_0^\infty e^{-zt} F^*(t) t^{q-(1/2)}\, dt \tag{2.9}$$

for some uniquely determined $F^* \in L_2(0, \infty)$. From the isometry (2.3) and (2.7), we thus obtain the desired inequality (1.2).

Next, in order to prove the equality statement, we recall that equality holds in (2.7) if and only if each f_j is expressible in the form $c_j K_{1/2}(z, \overline{u})$ for some constant c_j and for some point u such that $\operatorname{Re} u > 0$ and u is independent of j. See [**3**] for the case of the unit disc and $q = 1$. Note that the result [**3**] also implies the desired result for, in general, q for the present situation. See again [**1**, p. 361, Theorem II]. Hence,

$$f_j(z) = c_j K_{1/2}(z, \overline{u}) = c_j \int_0^\infty e^{-zt} e^{-\overline{u}t}\, dt, \tag{2.10}$$

and we thus see that each F_j is expressible in the desired form $c_j e^{-t\overline{u}}$.

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