PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 2, June 1984

NATURALLY REDUCTIVE METRICS OF NONPOSITIVE RICCI CURVATURE

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ABSTRACT. The main theorem states that every naturally reductive homogeneous Riemannian manifold of nonpositive Ricci curvature is symmetric. As a corollary, every noncompact naturally reductive Einstein manifold is symmetric.

A homogeneous space G/H is called naturally reductive if there exists a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ with $\operatorname{ad}(\mathfrak{h})\mathfrak{p} \subset \mathfrak{p}$ and

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p}.$$

The goal of this paper is to prove the following:

THEOREM. *Every naturally reductive Riemannian manifold of nonpositive Ricci curvature is symmetric.*

This strengthens a result of E. Deloff [**D**] asserting that every naturally reductive homogeneous manifold of nonpositive sectional curvature is symmetric. It also has the following consequence. A metric is called Einstein if there exists a constant E such that $\operatorname{Ric}(X, Y) = E\langle X, Y\rangle$; E is called the Einstein constant.

COROLLARY. *Every naturally reductive homogeneous Einstein manifold with nonpositive Einstein constant is symmetric.*

In particular, every noncompact naturally reductive Einstein manifold is symmetric. This is in sharp contrast to the compact case, where naturally reductive metrics provide a rich source of Einstein metrics [**DZ, WZ**].

To establish some preliminaries, let M be a connected homogeneous Riemannian manifold, G a transitive group of isometries of M and H the isotropy subgroup of G at a point $p \in M$. For convenience, we assume G acts effectively on M, i.e., only e acts as the identity transformation on M. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, and by $\mathfrak{p}$ an $\operatorname{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. M is naturally identified with the tangent space T_pM. Under this identification the Riemannian structure defines an $\operatorname{Ad}(H)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$. The metric is called naturally reductive (with respect to G and $\mathfrak{p}$) if $\langle [X, Y]_\mathfrak{p}, Z] + \langle Y, [X, Z]_\mathfrak{p}\rangle = 0$ for all $X, Y, Z \in \mathfrak{p}$, where $[X, Y]_\mathfrak{p}$ is the $\mathfrak{p}$ component of $[X, Y]$. Let

$$\bar{\mathfrak{g}} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}], \quad \bar{\mathfrak{h}} = \mathfrak{h} \cap \bar{\mathfrak{g}} \quad \text{and} \tag{1}$$
$$\bar{G} = \text{the subgroup of } G \text{ with Lie algebra } \bar{\mathfrak{g}}.$$

Received by the editors May 9, 1983.
1980 *Mathematics Subject Classification*. Primary 53C30.
[1]The second author was partially supported by a grant from the National Science Foundation.

Then $\bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$ and $\bar{G}$ acts transitively on M. By a theorem of Kostant, see e.g. [**DZ**, p. 4], if M is naturally reductive there exists a unique symmetric nondegenerate bilinear form Q on $\bar{\mathfrak{g}}$, invariant under $\mathrm{Ad}(\bar{G})$, such that $Q(\bar{\mathfrak{h}}, \mathfrak{p}) = 0$ and $Q_{|\mathfrak{p}}$ is equal to the given metric. Using Q one can express the Ricci curvature of the metric as follows [**WZ**]:

$$\text{(2)} \qquad \mathrm{Ric}(X, Y) = -\tfrac{1}{4}B_{\bar{\mathfrak{g}}}(X, Y) - \tfrac{1}{2}Q\left(C_{\chi, Q_{|\bar{\mathfrak{h}}}}(X), Y\right), \qquad X, Y \in \mathfrak{p},$$

where $B_{\bar{\mathfrak{g}}}$ is the Killing form of $\bar{\mathfrak{g}}$, χ the isotropy representation of the Lie algebra $\bar{\mathfrak{h}}$ on $\mathfrak{p}$, and $C_{\phi, g}$ is the Casimir operator of the orthogonal representation ϕ of $\mathfrak{h}$ w.r.t. a nondegenerate, symmetric, bilinear form g on $\mathfrak{h}$ invariant under $\mathrm{Ad}\, H$, i.e. $C_{\phi,g} = \sum_i \phi(X_i)\phi(Y_i)$ with $g(X_i, Y_j) = \delta_{ij}$. Note that Q need not be positive definite on $\bar{\mathfrak{g}}$ and hence $Q_{|\bar{\mathfrak{h}}}$ need not be positive definite.

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is called compactly embedded if $\mathrm{Ad}_G(K)$ is compact in $\mathrm{Ad}_G(G)$. If G/H is a Riemannian homogeneous space, then $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$. But H is only compact if G is closed in the full isometry group of $\langle\ ,\ \rangle$. A compactly embedded subalgebra is the direct sum of its center and semisimple ideal.

(3) Lemma. *Suppose G/H is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Let $\mathfrak{u}$ be a compactly embedded subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{u} \subset \mathfrak{g}$. Then, unless $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$ for some ideal $\mathfrak{t}$ in the center of $\mathfrak{g}$, there exists $X \in \mathfrak{p} \cap \mathfrak{u}$ with* $\mathrm{Ric}(X, X) > 0$.

Proof. Using the notation in (1) let $\bar{\mathfrak{u}} = \mathfrak{u} \cap \bar{\mathfrak{g}}$. Then $\bar{\mathfrak{h}} \subset \bar{\mathfrak{u}} \subset \bar{\mathfrak{g}}$ and $\bar{\mathfrak{u}}$ is compactly embedded in $\bar{\mathfrak{g}}$. Hence if $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$, $B_{\bar{\mathfrak{g}}}(X, X) \leqslant 0$ with equality iff X is in the center of $\bar{\mathfrak{g}}$. Hence the first term in (2) for $\mathrm{Ric}(X, X)$ is nonnegative, but the second term can have either sign.

Let $\bar{U}$ be a compact Lie group with Lie algebra $\bar{\mathfrak{u}}$, and $\bar{H} \subset \bar{U}$ the subgroup corresponding to $\bar{\mathfrak{h}}$. Then $Q_{|\bar{\mathfrak{u}}}$ induces a naturally reductive metric on $\bar{U}/\bar{H}$. If $\mathrm{Ric}(X, X) \leqslant 0$ for all $X \in \mathfrak{p} \cap \mathfrak{u}$, then this metric on $\bar{U}/\bar{H}$ also has nonpositive Ricci curvature since the second term in (2) is the same for both naturally reductive metrics and the first term is related by $B_{\bar{\mathfrak{g}}}(X, X) \leqslant B_{\bar{\mathfrak{u}}}(X, X)$ (since $\bar{\mathfrak{u}}$ is compactly embedded in $\bar{\mathfrak{g}}$).

$\bar{U}/\bar{H}$ might not be effective, but by dividing by a common normal subgroup we obtain an effective compact homogeneous space $\bar{U}'/\bar{H}'$ with a naturally reductive metric with $\mathrm{Ric} \leqslant 0$. To this metric we apply Bochner's theorem [**K**, p. 57], which states that every Killing vector field on a compact manifold with $\mathrm{Ric} \leqslant 0$ is parallel. Hence $\bar{H}'$ must be finite since a parallel Killing vector field cannot vanish anywhere. Therefore $\bar{\mathfrak{h}}$ is an ideal in $\bar{\mathfrak{u}}$ and the isotropy action is trivial on $\bar{\mathfrak{u}} \cap \mathfrak{p}$. Applying (2) to $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$ we see that $\mathrm{Ric}(X, X) = -\frac{1}{4}B_{\bar{\mathfrak{g}}}(X, X) > 0$ unless X is in the center of $\bar{\mathfrak{g}}$. Since $\bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$ and $\bar{\mathfrak{h}}$ an ideal in $\mathfrak{h}$, this implies thc lemma.

Remark. This lemma becomes false without the assumption that G/H is naturally reductive. For example $\mathrm{SL}(n, \mathbf{R})$ admits a left-invariant metric with negative Ricci curvature [**LM**].

Recall that a connected Lie group G admits a Levi decomposition $G = G_1 \cdot G_2$, where G_1 is a maximal connected semisimple subgroup, unique up to conjugacy, and G_2 is the solvable radical of G. The semisimple group G_1 admits an Iwasawa decomposition $G_1 = K \cdot S$, again unique up to conjugacy, where S is solvable, the Lie algebra $\mathfrak{k}$ of K is compactly embedded in $\mathfrak{g}_1$, and $K \cap S = \{e\}$. K is compact iff the center of G_1 is finite. Under any left-invariant Riemannian metric, G_1/K is a symmetric space of nonpositive curvature on which S acts simply transitively by isometries. The reader is referred to Helgason **[H]** for further details.

(4) LEMMA. *Let $M = G/H$ be naturally reductive with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and suppose M has nonpositive Ricci curvature. Shrink G if necessary so that the group $\overline{G}$ in* (1) *is equal to G. Then there exists a semisimple Levi factor G_1 of G and an Iwasawa decomposition $G_1 = K \cdot S$ such that $K \subset H$.*

PROOF. We first show that for suitable choices of decompositions $G = G_1 \cdot G_2$, $G_1 = K \cdot S$, there exists a compactly embedded subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ containing both $\mathfrak{h}$ and $\mathfrak{k}$.

Let G' be the full isometry group of M and H' its (compact) isotropy group. Since $G \subset G'$ we can choose Levi factors G_1 and G'_1 of G and G' and Iwasawa decompositions $G_1 = K \cdot S$ and $G'_1 = K' \cdot S'$ satisfying $G_1 \subset G'_1$, $K \subset K'$, $S \subset S'$. Note that H is compact iff G is closed in G' and K is compact iff G_1 has finite center. Since the claim involves only the Lie algebras, we may assume, after modding out a discrete central subgroup, if necessary, that G'_1 has finite center. Hence K' is compact and lies in a maximal compact subgroup U' of G'. Since all maximal compact subgroups are conjugate, there exists $x \in G'$ with $xH'x^{-1} \subset U'$. Since $G' = G \cdot H'$, we may choose x to lie in G. Letting $U = U' \cap G$, U contains both K and xHx^{-1} and has Lie algebra $\mathfrak{u}$ compactly embedded in $\mathfrak{g}$. Replacing each of G_1, K, S and U by their conjugates under x^{-1}, we have $H, K \subset U$ as desired.

By (3) $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$ for some ideal $\mathfrak{t}$ in the center of $\mathfrak{g}$. Since the adjoint representation of $\mathfrak{h}$ acts trivially on $\mathfrak{p} \cap \mathfrak{u}$, we have $\mathrm{Ric}(X, X) = -\frac{1}{4}B_{\mathfrak{g}}(X, X)$ for $X \in \mathfrak{p} \cap \mathfrak{u}$ by (2) and, hence, $\mathfrak{p} \cap \mathfrak{u} \subset \mathfrak{z}(\mathfrak{g})$, which implies $\mathfrak{p} \cap \mathfrak{u} = \mathfrak{t}$. Hence $\mathfrak{t}$ and $\mathfrak{h}$ are orthogonal with respect to Q. Since Q is $\mathrm{Ad}(G)$-invariant and $\mathfrak{g}_1$ is semisimple, $\mathfrak{g}_1$, and hence $\mathfrak{k}$, must also be orthogonal to $\mathfrak{t}$ with respect to Q. But this implies $\mathfrak{k} \subset \mathfrak{h}$.

PROOF OF THE THEOREM. If G_1 is a Levi factor of G we can write $G_1 = G_{\mathrm{nc}}G_{\mathrm{c}}$ where G_{nc} and G_{c}, the noncompact and compact parts of G_1, are the products of all noncompact, respectively compact, simple normal subgroups of G_1. Then $K = (K \cap G_{\mathrm{nc}}) \cdot G_{\mathrm{c}}$ and $S \subset G_{\mathrm{nc}}$. Similarly for the full isometry group G' we write $G'_1 = G'_{\mathrm{nc}} \cdot G'_{\mathrm{c}}$. To finish the proof, we will use the following result from **[G]**:

(5) Let G/H be naturally reductive. Then there exists a nilpotent normal subgroup N of G such that $G_1 \cdot N$ acts transitively on M, G_{nc} commutes with N, and $G_{\mathrm{nc}} = G'_{\mathrm{nc}}$.

Since, by (4), $K \subset H$ and since the center of G_{nc} lies in $K \cap G_{\mathrm{nc}} \subset H$, the effectiveness of G/H implies that G_{nc} has trivial center and, hence, $K \cap G_{\mathrm{nc}}$ and K are compact. Since $G_1 \cdot N$ acts transitively, (4) also implies that $G_{\mathrm{nc}} \cdot N$ and $S \cdot N$

act transitively. We first claim that $H \cap (G_{nc}N) = (K \cap G_{nc})(H \cap N)$ and, therefore, that $H \cap (S \cdot N) \subset N$. Since $N \subset G_2'$, $G_{nc}G_2'$ also acts transitively, and since $G_{nc} = G_{nc}'$, $G_{nc}G_2'$ is closed in G'. Hence the isotropy subgroup L of $G_{nc}G_2'$ is compact. Since G_{nc} has no center, $G_{nc} \cap G_2' = \{e\}$. The projection of L onto G_{nc} is compact and contains the maximal compact subgroup $K \cap G_{nc}$. Hence the projection is $K \cap G_{nc}$ and we obtain $L = (K \cap G_{nc})(L \cap G_2')$; and hence

$$H \cap (G_{nc}N) = L \cap (G_{nc}N) = (K \cap G_{nc})(L \cap N) = (K \cap G_{nc})(H \cap N).$$

Now $\mathfrak{h} \cap \mathfrak{n} = \{0\}$ since $\mathrm{ad}_{\mathfrak{g}}\, \mathfrak{h}$ contains no nilpotent operators and, hence, $\mathfrak{h} \cap (\mathfrak{s} + \mathfrak{n}) = \{0\}$, i.e., SN acts almost simply transitively on M. Under the identification of $\mathfrak{s} + \mathfrak{n}$ with the tangent space T_pM, the isotropy action of $K \cap G_{nc} \subset H$ is trivial on $\mathfrak{n}$ and acts on $\mathfrak{s}$ without any trivial factors. Hence $\mathfrak{s}$ and $\mathfrak{n}$ are orthogonal w.r.t. the Riemannian metric and M is the Riemannian direct product $S \times N/N \cap H$. The metric on $S = G_{nc}/K$ is left G_{nc}-invariant and hence symmetric. N may be given a left-invariant metric of nonpositive Ricci curvature so that N is a Riemannian covering of $N/N \cap H$. But a left-invariant metric on a nilpotent Lie group is either flat or else has Ricci curvatures of both signs (see [**M**, Theorem 2.4]). Thus the metric on N is flat and hence M is symmetric.

Added in proof. There is an error in the second paragraph of the proof of Lemma 4; the discrete center of G_1' need not be closed in G'. For a different proof of the existence of $\mathfrak{u}$, see [**G–W**, Remark 3.4].

References

[**DZ**] J. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*, Mem. Amer. Math. Soc., No. 215 (1979).

[**D**] E. D. Deloff, *Naturally reductive metrics and metrics with volume preserving geodesic symmetries on NC algebras*, Thesis, Rutgers, 1979.

[**G**] C. Gordon, *Naturally reductive Riemannian manifolds*, preprint 1984.

[**G–W**] C. Gordon and E. N. Wilson, *The fine structure of transitive Riemannian isometry groups*. I, preprint 1984.

[**H**] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[**K**] S. Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, Berlin and New York, 1972.

[**LM**] M. L. Leite and I. D. Miatello, *Metrics of negative Ricci curvature on* $\mathrm{SL}(n, \mathbf{R})$, $n \geqslant 3$, J. Differential Geom. **17** (1982), 635–641.

[**M**] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 243–329.

[**WZ**] M. Wang and W. Ziller, *On normal homogeneous Einstein metrics*, preprint 1984.

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