NATURALLY REDUCTIVE METRICS OF NONPOSITIVE RICCI CURVATURE

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Abstract. The main theorem states that every naturally reductive homogeneous Riemannian manifold of nonpositive Ricci curvature is symmetric. As a corollary, every noncompact naturally reductive Einstein manifold is symmetric.

A homogeneous space \( G/H \) is called naturally reductive if there exists a decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{p} \) with \( \text{ad}(\mathfrak{h})\mathfrak{p} \subseteq \mathfrak{p} \) and

\[
\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p}.
\]

The goal of this paper is to prove the following:

**THEOREM.** Every naturally reductive Riemannian manifold of nonpositive Ricci curvature is symmetric.

This strengthens a result of E. Deloff [D] asserting that every naturally reductive homogeneous manifold of nonpositive sectional curvature is symmetric. It also has the following consequence. A metric is called Einstein if there exists a constant \( E \) such that \( \text{Ric}(X, Y) = E \langle X, Y \rangle \); \( E \) is called the Einstein constant.

**COROLLARY.** Every naturally reductive homogeneous Einstein manifold with nonpositive Einstein constant is symmetric.

In particular, every noncompact naturally reductive Einstein manifold is symmetric. This is in sharp contrast to the compact case, where naturally reductive metrics provide a rich source of Einstein metrics [DZ, WZ].

To establish some preliminaries, let \( M \) be a connected homogeneous Riemannian manifold, \( G \) a transitive group of isometries of \( M \) and \( H \) the isotropy subgroup of \( G \) at a point \( p \in M \). For convenience, we assume \( G \) acts effectively on \( M \), i.e., only \( e \) acts as the identity transformation on \( M \). Denote by \( \mathfrak{g} \) and \( \mathfrak{h} \) the Lie algebras of \( G \) and \( H \), and by \( \mathfrak{p} \) an \( \text{Ad}(H) \)-invariant complement of \( \mathfrak{h} \) in \( \mathfrak{g} \). \( M \) is naturally identified with the tangent space \( T_p M \). Under this identification the Riemannian structure defines an \( \text{Ad}(H) \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{p} \). The metric is called naturally reductive (with respect to \( G \) and \( \mathfrak{p} \)) if \( \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \) for all \( X, Y, Z \in \mathfrak{p} \), where \( [X, Y]_\mathfrak{p} \) is the \( \mathfrak{p} \)-component of \( [X, Y] \). Let

\[
\begin{align*}
\mathfrak{g} &= \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}], & \mathfrak{h} &= \mathfrak{h} \cap \mathfrak{g} \quad \text{and} \\
\bar{G} &= \text{the subgroup of } G \text{ with Lie algebra } \mathfrak{g}.
\end{align*}
\]

Received by the editors May 9, 1983.

1980 Mathematics Subject Classification. Primary 53C30.

The second author was partially supported by a grant from the National Science Foundation.

\( \odot 1984 \) American Mathematical Society

0002-9939/84 $1.00 + $.25 per page
Then \( \mathfrak{g} \) is an ideal in \( \mathfrak{g} \) and \( \mathfrak{g} \) acts transitively on \( M \). By a theorem of Kostant, see e.g. [DZ, p. 4], if \( M \) is naturally reductive there exists a unique symmetric nondegenerate bilinear form \( Q \) on \( \mathfrak{g} \), invariant under \( \text{Ad}(\mathfrak{g}) \), such that \( Q(\mathfrak{h}, \mathfrak{v}) = 0 \) and \( Q^\mathfrak{u} \) is equal to the given metric. Using \( Q \) one can express the Ricci curvature of the metric as follows [WZ]:

\[
(2) \quad \text{Ric}(X, Y) = -\frac{1}{2} B_\mathfrak{h}(X, Y) - \frac{1}{4} Q\left(C_X Q^\mathfrak{u}(X), Y\right), \quad X, Y \in \mathfrak{v},
\]

where \( B_\mathfrak{h} \) is the Killing form of \( \mathfrak{h} \), \( \chi \) the isotropy representation of the Lie algebra \( \mathfrak{h} \) on \( \mathfrak{v} \), and \( C_{\phi,\mathfrak{g}} \) is the Casimir operator of the orthogonal representation \( \phi \) of \( \mathfrak{h} \) w.r.t. a nondegenerate, symmetric, bilinear form \( g \) on \( \mathfrak{h} \) invariant under \( \text{Ad} H \), i.e. \( C_{\phi,\mathfrak{g}} = \Sigma_\phi(X_i)\phi(Y_i) \) with \( g(X_i, Y_i) = \delta_{ij} \). Note that \( Q \) need not be positive definite on \( \mathfrak{g} \) and hence \( Q^\mathfrak{u} \) need not be positive definite.

A subalgebra \( \mathfrak{f} \subset \mathfrak{g} \) is called compactly embedded if \( \text{Ad}_\mathfrak{f}(K) \) is compact in \( \text{Ad}_\mathfrak{G}(G) \). If \( G/H \) is a Riemannian homogeneous space, then \( \mathfrak{h} \) is compactly embedded in \( \mathfrak{g} \). But \( H \) is only compact if \( G \) is closed in the full isometry group of \( \langle \cdot, \cdot \rangle \). A compactly embedded subalgebra is the direct sum of its center and semisimple ideal.

(3) **Lemma.** Suppose \( G/H \) is naturally reductive with respect to the decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v} \). Let \( \mathfrak{u} \) be a compactly embedded subalgebra of \( \mathfrak{g} \) with \( \mathfrak{h} \subset \mathfrak{u} \subset \mathfrak{g} \). Then, unless \( \mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t} \) for some ideal \( \mathfrak{t} \) in the center of \( \mathfrak{g} \), there exists \( X \in \mathfrak{v} \cap \mathfrak{u} \) with \( \text{Ric}(X, X) > 0 \).

**Proof.** Using the notation in (1) let \( \bar{\mathfrak{u}} = \mathfrak{u} \cap \bar{\mathfrak{g}} \). Then \( \mathfrak{h} \subset \bar{\mathfrak{u}} \subset \bar{\mathfrak{g}} \) and \( \bar{\mathfrak{u}} \) is compactly embedded in \( \bar{\mathfrak{g}} \). Hence if \( X \in \bar{\mathfrak{u}} \cap \mathfrak{v} \), \( B_\mathfrak{h}(X, X) \leq 0 \) with equality iff \( X \) is in the center of \( \bar{\mathfrak{g}} \). Hence the first term in (2) for \( \text{Ric}(X, X) \) is nonnegative, but the second term can have either sign.

Let \( \bar{U} \) be a compact Lie group with Lie algebra \( \bar{\mathfrak{u}} \), and \( \bar{H} \subset \bar{U} \) the subgroup corresponding to \( \bar{\mathfrak{h}} \). Then \( Q_{|\bar{\mathfrak{u}}} \) induces a naturally reductive metric on \( \bar{U}/\bar{H} \). If \( \text{Ric}(X, X) \leq 0 \) for all \( X \in \mathfrak{v} \cap \mathfrak{u} \), then this metric on \( \bar{U}/\bar{H} \) also has nonpositive Ricci curvature since the second term in (2) is the same for both naturally reductive metrics and the first term is related by \( B_\mathfrak{h}(X, X) \leq B_\mathfrak{u}(X, X) \) (since \( \bar{\mathfrak{u}} \) is compactly embedded in \( \bar{\mathfrak{g}} \)).

\( \bar{U}/\bar{H} \) might not be effective, but by dividing by a common normal subgroup we obtain an effective compact homogeneous space \( \bar{U}/\bar{H} \) with a naturally reductive metric with \( \text{Ric} \leq 0 \). To this metric we apply Bochner's theorem [K, p. 57], which states that every Killing vector field on a compact manifold with \( \text{Ric} \leq 0 \) is parallel. Hence \( \bar{H} \) must be finite since a parallel Killing vector field cannot vanish anywhere. Therefore \( \bar{\mathfrak{h}} \) is an ideal in \( \bar{\mathfrak{u}} \) and the isotropy action is trivial on \( \bar{\mathfrak{u}} \cap \mathfrak{v} \). Applying (2) to \( X \in \bar{\mathfrak{u}} \cap \mathfrak{v} \) we see that \( \text{Ric}(X, X) = -\frac{1}{4} B_\mathfrak{h}(X, X) > 0 \) unless \( X \) is in the center of \( \bar{\mathfrak{g}} \). Since \( \bar{\mathfrak{g}} \) is an ideal in \( \mathfrak{g} \) and \( \bar{\mathfrak{h}} \) an ideal in \( \mathfrak{h} \), this implies the lemma.

**Remark.** This lemma becomes false without the assumption that \( G/H \) is naturally reductive. For example \( \text{SL}(n, \mathbb{R}) \) admits a left-invariant metric with negative Ricci curvature [LM].
Recall that a connected Lie group $G$ admits a Levi decomposition $G = G_1 \cdot G_2$, where $G_1$ is a maximal connected semisimple subgroup, unique up to conjugacy, and $G_2$ is the solvable radical of $G$. The semisimple group $G_1$ admits an Iwasawa decomposition $G_1 = K \cdot S$, again unique up to conjugacy, where $S$ is solvable, the Lie algebra $\mathfrak{k}$ of $K$ is compactly embedded in $\mathfrak{g}_1$, and $K \cap S = \{e\}$. $K$ is compact iff the center of $G_1$ is finite. Under any left-invariant Riemannian metric, $G_1/K$ is a symmetric space of nonpositive curvature on which $S$ acts simply transitively by isometries. The reader is referred to Helgason [H] for further details.

(4) Lemma. Let $M = G/H$ be naturally reductive with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ and suppose $M$ has nonpositive Ricci curvature. Shrink $G$ if necessary so that the group $\bar{G}$ in (1) is equal to $G$. Then there exists a semisimple Levi factor $G_1$ of $G$ and an Iwasawa decomposition $G_1 = K \cdot S$ such that $K \subset H$.

Proof. We first show that for suitable choices of decompositions $G = G_1 \cdot G_2$, $G_1 = K \cdot S$, there exists a compactly embedded subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ containing both $\mathfrak{h}$ and $\mathfrak{k}$.

Let $G'$ be the full isometry group of $M$ and $H'$ its (compact) isotropy group. Since $G < G'$ we can choose Levi factors $G_1$ and $G'_1$ of $G$ and $G'$ and Iwasawa decompositions $G_1 = K \cdot S$ and $G'_1 = K' \cdot S'$ satisfying $G_1 \subset G'_1$, $K \subset K'$, $S \subset S'$. Note that $H$ is compact iff $G$ is closed in $G'$ and $K$ is compact iff $G_1$ has finite center. Since the claim involves only the Lie algebras, we may assume, after modding out a discrete central subgroup, if necessary, that $G_1$ has finite center. Hence $K'$ is compact and lies in a maximal compact subgroup $U'$ of $G'$. Since all maximal compact subgroups are conjugate, there exists $x \in G'$ with $xH'x^{-1} \subset U'$. Since $G' = G \cdot H'$, we may choose $x$ to lie in $G$. Letting $U = U' \cap G$, $U$ contains both $K$ and $xHx^{-1}$ and has Lie algebra $\mathfrak{u}$ compactly embedded in $\mathfrak{g}$. Replacing each of $G_1$, $K$, $S$ and $U$ by their conjugates under $x^{-1}$ we have $H$, $K \subset U$ as desired.

By (3) $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$ for some ideal $\mathfrak{t}$ in the center of $\mathfrak{g}$. Since the adjoint representation of $\mathfrak{h}$ acts trivially on $\mathfrak{v} \cap \mathfrak{u}$, we have $\text{Ric}(X, X) = -\frac{1}{2}B_\mathfrak{h}(X, X)$ for $X \in \mathfrak{v} \cap \mathfrak{u}$ by (2) and, hence, $\mathfrak{v} \cap \mathfrak{u} \subset \mathfrak{g}(\mathfrak{h})$, which implies $\mathfrak{v} \cap \mathfrak{u} = \mathfrak{t}$. Hence $\mathfrak{t}$ and $\mathfrak{h}$ are orthogonal with respect to $Q$. Since $Q$ is $\text{Ad}(G)$-invariant and $\mathfrak{g}_1$ is semisimple, $\mathfrak{g}_1$, and hence $\mathfrak{t}$, must also be orthogonal to $\mathfrak{t}$ with respect to $Q$. But this implies $\mathfrak{t} \subset \mathfrak{h}$.

Proof of the Theorem. If $G_1$ is a Levi factor of $G$ we can write $G_1 = G_\text{nc} \cdot G_\text{c}$ where $G_\text{nc}$ and $G_\text{c}$, the noncompact and compact parts of $G_1$, are the products of all noncompact, respectively compact, simple normal subgroups of $G_1$. Then $K = (K \cap G_\text{nc}) \cdot G_\text{c}$ and $S \subset G_\text{nc}$. Similarly for the full isometry group $G'$ we write $G'_1 = G'_\text{nc} \cdot G'_\text{c}$. To finish the proof, we will use the following result from [G]:

(5) Let $G/H$ be naturally reductive. Then there exists a nilpotent normal subgroup $N$ of $G$ such that $G_1 \cdot N$ acts transitively on $M$, $G_\text{nc}$ commutes with $N$, and $G_\text{nc} = G_\text{nc}$.

Since, by (4), $K \subset H$ and since the center of $G_\text{nc}$ lies in $K \cap G_\text{nc} \subset H$, the effectiveness of $G/H$ implies that $G_\text{nc}$ has trivial center and, hence, $K \cap G_\text{nc}$ and $K$ are compact. Since $G_1 \cdot N$ acts transitively, (4) also implies that $G_\text{nc} \cdot N$ and $S \cdot N$
act transitively. We first claim that $H \cap (G_{nc} N) = (K \cap G_{nc})(H \cap N)$ and, therefore, that $H \cap (S \cdot N) \subset N$. Since $N \subset G'_2$, $G_{nc} G'_2$ also acts transitively, and since $G_{nc} = G'_n$, $G_{nc} G'_2$ is closed in $G'$. Hence the isotropy subgroup $L$ of $G_{nc} G'_2$ is compact. Since $G_{nc}$ has no center, $G_{nc} \cap G'_2 = \{e\}$. The projection of $L$ onto $G_{nc}$ is compact and contains the maximal compact subgroup $K \cap G_{nc}$. Hence the projection is $K \cap G_{nc}$ and we obtain $L = (K \cap G_{nc})(L \cap G'_2)$; and hence

$$H \cap (G_{nc} N) = L \cap (G_{nc} N) = (K \cap G_{nc})(L \cap N) = (K \cap G_{nc})(H \cap N).$$

Now $\mathfrak{h} \cap \mathfrak{n} = \{0\}$ since $\mathrm{ad}_{\mathfrak{a}} \mathfrak{h}$ contains no nilpotent operators and, hence, $\mathfrak{h} \cap (\mathfrak{s} + \mathfrak{n}) = \{0\}$, i.e., $SN$ acts almost simply transitively on $M$. Under the identification of $\mathfrak{s} + \mathfrak{n}$ with the tangent space $T_p M$, the isotropy action of $K \cap G_{nc} \subset H$ is trivial on $\mathfrak{n}$ and acts on $\mathfrak{s}$ without any trivial factors. Hence $\mathfrak{s}$ and $\mathfrak{n}$ are orthogonal w.r.t. the Riemannian metric and $M$ is the Riemannian direct product $S \times N/N \cap H$. The metric on $S = G_{nc}/K$ is left $G_{nc}$-invariant and hence symmetric. $N$ may be given a left-invariant metric of nonpositive Ricci curvature so that $N$ is a Riemannian covering of $N/N \cap H$. But a left-invariant metric on a nilpotent Lie group is either flat or else has Ricci curvatures of both signs (see [M, Theorem 2.4]). Thus the metric on $N$ is flat and hence $M$ is symmetric.

**ADDED IN PROOF.** There is an error in the second paragraph of the proof of Lemma 4; the discrete center of $G'_1$ need not be closed in $G'$. For a different proof of the existence of $u$, see [G–W, Remark 3.4].

**References**


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