

NATURALLY REDUCTIVE METRICS OF NONPOSITIVE RICCI CURVATURE

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ABSTRACT. The main theorem states that every naturally reductive homogeneous Riemannian manifold of nonpositive Ricci curvature is symmetric. As a corollary, every noncompact naturally reductive Einstein manifold is symmetric.

A homogeneous space G/H is called naturally reductive if there exists a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ with $\text{ad}(\mathfrak{h})\mathfrak{p} \subset \mathfrak{p}$ and

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \quad \text{for all } X, Y, Z \in \mathfrak{p}.$$

The goal of this paper is to prove the following:

THEOREM. *Every naturally reductive Riemannian manifold of nonpositive Ricci curvature is symmetric.*

This strengthens a result of E. Deloff [D] asserting that every naturally reductive homogeneous manifold of nonpositive sectional curvature is symmetric. It also has the following consequence. A metric is called Einstein if there exists a constant E such that $\text{Ric}(X, Y) = E\langle X, Y \rangle$; E is called the Einstein constant.

COROLLARY. *Every naturally reductive homogeneous Einstein manifold with nonpositive Einstein constant is symmetric.*

In particular, every noncompact naturally reductive Einstein manifold is symmetric. This is in sharp contrast to the compact case, where naturally reductive metrics provide a rich source of Einstein metrics [DZ, WZ].

To establish some preliminaries, let M be a connected homogeneous Riemannian manifold, G a transitive group of isometries of M and H the isotropy subgroup of G at a point $p \in M$. For convenience, we assume G acts effectively on M , i.e., only e acts as the identity transformation on M . Denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of G and H , and by \mathfrak{p} an $\text{Ad}(H)$ -invariant complement of \mathfrak{h} in \mathfrak{g} . M is naturally identified with the tangent space $T_p M$. Under this identification the Riemannian structure defines an $\text{Ad}(H)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} . The metric is called naturally reductive (with respect to G and \mathfrak{p}) if $\langle [X, Y]_{\mathfrak{p}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{p}} \rangle = 0$ for all $X, Y, Z \in \mathfrak{p}$, where $[X, Y]_{\mathfrak{p}}$ is the \mathfrak{p} component of $[X, Y]$. Let

$$(1) \quad \begin{aligned} \bar{\mathfrak{g}} &= \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}], & \bar{\mathfrak{h}} &= \mathfrak{h} \cap \bar{\mathfrak{g}} \quad \text{and} \\ \bar{G} &= \text{the subgroup of } G \text{ with Lie algebra } \bar{\mathfrak{g}}. \end{aligned}$$

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Then $\bar{\mathfrak{g}}$ is an ideal in \mathfrak{g} and \bar{G} acts transitively on M . By a theorem of Kostant, see e.g. [DZ, p. 4], if M is naturally reductive there exists a unique symmetric nondegenerate bilinear form Q on $\bar{\mathfrak{g}}$, invariant under $\text{Ad}(\bar{G})$, such that $Q(\bar{\mathfrak{h}}, \mathfrak{p}) = 0$ and $Q|_{\mathfrak{p}}$ is equal to the given metric. Using Q one can express the Ricci curvature of the metric as follows [WZ]:

$$(2) \quad \text{Ric}(X, Y) = -\frac{1}{4}B_{\bar{\mathfrak{g}}}(X, Y) - \frac{1}{2}Q(C_{\chi, Q|_{\bar{\mathfrak{h}}}}(X), Y), \quad X, Y \in \mathfrak{p},$$

where $B_{\bar{\mathfrak{g}}}$ is the Killing form of $\bar{\mathfrak{g}}$, χ the isotropy representation of the Lie algebra $\bar{\mathfrak{h}}$ on \mathfrak{p} , and $C_{\phi, g}$ is the Casimir operator of the orthogonal representation ϕ of \mathfrak{h} w.r.t. a nondegenerate, symmetric, bilinear form g on \mathfrak{h} invariant under $\text{Ad } H$, i.e. $C_{\phi, g} = \sum_i \phi(X_i)\phi(Y_i)$ with $g(X_i, Y_j) = \delta_{ij}$. Note that Q need not be positive definite on $\bar{\mathfrak{g}}$ and hence $Q|_{\bar{\mathfrak{h}}}$ need not be positive definite.

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is called compactly embedded if $\text{Ad}_G(K)$ is compact in $\text{Ad}_G(G)$. If G/H is a Riemannian homogeneous space, then \mathfrak{h} is compactly embedded in \mathfrak{g} . But H is only compact if G is closed in the full isometry group of $\langle \cdot, \cdot \rangle$. A compactly embedded subalgebra is the direct sum of its center and semisimple ideal.

(3) LEMMA. *Suppose G/H is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Let \mathfrak{u} be a compactly embedded subalgebra of \mathfrak{g} with $\mathfrak{h} \subset \mathfrak{u} \subset \mathfrak{g}$. Then, unless $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$ for some ideal \mathfrak{t} in the center of \mathfrak{g} , there exists $X \in \mathfrak{p} \cap \mathfrak{u}$ with $\text{Ric}(X, X) > 0$.*

PROOF. Using the notation in (1) let $\bar{\mathfrak{u}} = \mathfrak{u} \cap \bar{\mathfrak{g}}$. Then $\bar{\mathfrak{h}} \subset \bar{\mathfrak{u}} \subset \bar{\mathfrak{g}}$ and $\bar{\mathfrak{u}}$ is compactly embedded in $\bar{\mathfrak{g}}$. Hence if $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$, $B_{\bar{\mathfrak{g}}}(X, X) \leq 0$ with equality iff X is in the center of $\bar{\mathfrak{g}}$. Hence the first term in (2) for $\text{Ric}(X, X)$ is nonnegative, but the second term can have either sign.

Let \bar{U} be a compact Lie group with Lie algebra $\bar{\mathfrak{u}}$, and $\bar{H} \subset \bar{U}$ the subgroup corresponding to $\bar{\mathfrak{h}}$. Then $Q|_{\bar{\mathfrak{u}}}$ induces a naturally reductive metric on \bar{U}/\bar{H} . If $\text{Ric}(X, X) \leq 0$ for all $X \in \mathfrak{p} \cap \mathfrak{u}$, then this metric on \bar{U}/\bar{H} also has nonpositive Ricci curvature since the second term in (2) is the same for both naturally reductive metrics and the first term is related by $B_{\bar{\mathfrak{g}}}(X, X) \leq B_{\bar{\mathfrak{u}}}(X, X)$ (since $\bar{\mathfrak{u}}$ is compactly embedded in $\bar{\mathfrak{g}}$).

\bar{U}/\bar{H} might not be effective, but by dividing by a common normal subgroup we obtain an effective compact homogeneous space \bar{U}'/\bar{H}' with a naturally reductive metric with $\text{Ric} \leq 0$. To this metric we apply Bochner's theorem [K, p. 57], which states that every Killing vector field on a compact manifold with $\text{Ric} \leq 0$ is parallel. Hence \bar{H}' must be finite since a parallel Killing vector field cannot vanish anywhere. Therefore $\bar{\mathfrak{h}}$ is an ideal in $\bar{\mathfrak{u}}$ and the isotropy action is trivial on $\bar{\mathfrak{u}} \cap \mathfrak{p}$. Applying (2) to $X \in \bar{\mathfrak{u}} \cap \mathfrak{p}$ we see that $\text{Ric}(X, X) = -\frac{1}{4}B_{\bar{\mathfrak{g}}}(X, X) > 0$ unless X is in the center of $\bar{\mathfrak{g}}$. Since $\bar{\mathfrak{g}}$ is an ideal in \mathfrak{g} and $\bar{\mathfrak{h}}$ an ideal in \mathfrak{h} , this implies the lemma.

REMARK. This lemma becomes false without the assumption that G/H is naturally reductive. For example $\text{SL}(n, \mathbf{R})$ admits a left-invariant metric with negative Ricci curvature [LM].

Recall that a connected Lie group G admits a Levi decomposition $G = G_1 \cdot G_2$, where G_1 is a maximal connected semisimple subgroup, unique up to conjugacy, and G_2 is the solvable radical of G . The semisimple group G_1 admits an Iwasawa decomposition $G_1 = K \cdot S$, again unique up to conjugacy, where S is solvable, the Lie algebra \mathfrak{k} of K is compactly embedded in \mathfrak{g}_1 , and $K \cap S = \{e\}$. K is compact iff the center of G_1 is finite. Under any left-invariant Riemannian metric, G_1/K is a symmetric space of nonpositive curvature on which S acts simply transitively by isometries. The reader is referred to Helgason [H] for further details.

(4) LEMMA. *Let $M = G/H$ be naturally reductive with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and suppose M has nonpositive Ricci curvature. Shrink G if necessary so that the group \bar{G} in (1) is equal to G . Then there exists a semisimple Levi factor G_1 of G and an Iwasawa decomposition $G_1 = K \cdot S$ such that $K \subset H$.*

PROOF. We first show that for suitable choices of decompositions $G = G_1 \cdot G_2$, $G_1 = K \cdot S$, there exists a compactly embedded subalgebra \mathfrak{u} of \mathfrak{g} containing both \mathfrak{h} and \mathfrak{k} .

Let G' be the full isometry group of M and H' its (compact) isotropy group. Since $G \subset G'$ we can choose Levi factors G_1 and G'_1 of G and G' and Iwasawa decompositions $G_1 = K \cdot S$ and $G'_1 = K' \cdot S'$ satisfying $G_1 \subset G'_1$, $K \subset K'$, $S \subset S'$. Note that H is compact iff G is closed in G' and K is compact iff G_1 has finite center. Since the claim involves only the Lie algebras, we may assume, after modding out a discrete central subgroup, if necessary, that G'_1 has finite center. Hence K' is compact and lies in a maximal compact subgroup U' of G' . Since all maximal compact subgroups are conjugate, there exists $x \in G'$ with $xH'x^{-1} \subset U'$. Since $G' = G \cdot H'$, we may choose x to lie in G . Letting $U = U' \cap G$, U contains both K and xHx^{-1} and has Lie algebra \mathfrak{u} compactly embedded in \mathfrak{g} . Replacing each of G_1 , K , S and U by their conjugates under x^{-1} , we have $H, K \subset U$ as desired.

By (3) $\mathfrak{u} = \mathfrak{h} \oplus \mathfrak{t}$ for some ideal \mathfrak{t} in the center of \mathfrak{g} . Since the adjoint representation of \mathfrak{h} acts trivially on $\mathfrak{p} \cap \mathfrak{u}$, we have $\text{Ric}(X, X) = -\frac{1}{4}B_{\mathfrak{g}}(X, X)$ for $X \in \mathfrak{p} \cap \mathfrak{u}$ by (2) and, hence, $\mathfrak{p} \cap \mathfrak{u} \subset \mathfrak{z}(\mathfrak{g})$, which implies $\mathfrak{p} \cap \mathfrak{u} = \mathfrak{t}$. Hence \mathfrak{t} and \mathfrak{h} are orthogonal with respect to Q . Since Q is $\text{Ad}(G)$ -invariant and \mathfrak{g}_1 is semisimple, \mathfrak{g}_1 , and hence \mathfrak{k} , must also be orthogonal to \mathfrak{t} with respect to Q . But this implies $\mathfrak{k} \subset \mathfrak{h}$.

PROOF OF THE THEOREM. If G_1 is a Levi factor of G we can write $G_1 = G_{nc}G_c$ where G_{nc} and G_c , the noncompact and compact parts of G_1 , are the products of all noncompact, respectively compact, simple normal subgroups of G_1 . Then $K = (K \cap G_{nc}) \cdot G_c$ and $S \subset G_{nc}$. Similarly for the full isometry group G' we write $G'_1 = G'_{nc} \cdot G'_c$. To finish the proof, we will use the following result from [G]:

(5) Let G/H be naturally reductive. Then there exists a nilpotent normal subgroup N of G such that $G_1 \cdot N$ acts transitively on M , G_{nc} commutes with N , and $G_{nc} = G'_{nc}$.

Since, by (4), $K \subset H$ and since the center of G_{nc} lies in $K \cap G_{nc} \subset H$, the effectiveness of G/H implies that G_{nc} has trivial center and, hence, $K \cap G_{nc}$ and K are compact. Since $G_1 \cdot N$ acts transitively, (4) also implies that $G_{nc} \cdot N$ and $S \cdot N$

act transitively. We first claim that $H \cap (G_{nc}N) = (K \cap G_{nc})(H \cap N)$ and, therefore, that $H \cap (S \cdot N) \subset N$. Since $N \subset G'_2$, $G_{nc}G'_2$ also acts transitively, and since $G_{nc} = G'_{nc}$, $G_{nc}G'_2$ is closed in G' . Hence the isotropy subgroup L of $G_{nc}G'_2$ is compact. Since G_{nc} has no center, $G_{nc} \cap G'_2 = \{e\}$. The projection of L onto G_{nc} is compact and contains the maximal compact subgroup $K \cap G_{nc}$. Hence the projection is $K \cap G_{nc}$ and we obtain $L = (K \cap G_{nc})(L \cap G'_2)$; and hence

$$H \cap (G_{nc}N) = L \cap (G_{nc}N) = (K \cap G_{nc})(L \cap N) = (K \cap G_{nc})(H \cap N).$$

Now $\mathfrak{h} \cap \mathfrak{n} = \{0\}$ since $\text{ad}_{\mathfrak{g}} \mathfrak{h}$ contains no nilpotent operators and, hence, $\mathfrak{h} \cap (\mathfrak{s} + \mathfrak{n}) = \{0\}$, i.e., SN acts almost simply transitively on M . Under the identification of $\mathfrak{s} + \mathfrak{n}$ with the tangent space $T_p M$, the isotropy action of $K \cap G_{nc} \subset H$ is trivial on \mathfrak{n} and acts on \mathfrak{s} without any trivial factors. Hence \mathfrak{s} and \mathfrak{n} are orthogonal w.r.t. the Riemannian metric and M is the Riemannian direct product $S \times N/N \cap H$. The metric on $S = G_{nc}/K$ is left G_{nc} -invariant and hence symmetric. N may be given a left-invariant metric of nonpositive Ricci curvature so that N is a Riemannian covering of $N/N \cap H$. But a left-invariant metric on a nilpotent Lie group is either flat or else has Ricci curvatures of both signs (see [M, Theorem 2.4]). Thus the metric on N is flat and hence M is symmetric.

ADDED IN PROOF. There is an error in the second paragraph of the proof of Lemma 4; the discrete center of G'_1 need not be closed in G' . For a different proof of the existence of \mathfrak{u} , see [G-W, Remark 3.4].

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