THE EVENS-KAHN FORMULA
FOR THE TOTAL STIEFEL-WHITNEY CLASS
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ABSTRACT. Let $G(X)$ denote the (augmented) multiplicative group of classical cohomology ring of a space $X$, with coefficients in $\mathbb{Z}/2$. The (augmented) total Stiefel-Whitney class is a natural homomorphism $w: KO(X) \to G(X)$. We show that the functor $G(\ )$ possesses a 'transfer homomorphism' for double coverings such that $w$ commutes with the transfer. This is related to a question of G. Segal. As a special case, we obtain a formula for the total Stiefel-Whitney class of a representation of a finite group induced from a (real) representation of a subgroup of index 2, which is analogous to the one obtained by Evens and Kahn for the total Chern class.

1. Introduction. In [4] L. Evens and D. Kahn obtained an interesting formula for the total Chern class of the representation of a finite group induced from a (complex) representation of a subgroup of prime index $p$ (generalizing an earlier formula due to Evens [3]). The purpose of this note is to prove an analogous formula for the total Stiefel-Whitney class of the representation of a finite group induced from a (real) representation of a subgroup of index two. In fact, we formulate our statement in the context of homotopy theory, thus obtaining an interesting reformulation of a question of G. B. Segal [9].

Let $X$ be a CW-complex and let $G(X)$ denote the group of units of the commutative ring $\prod_{i \geq 0} H^i(X; Z/2)$. Let

$$\hat{G}(X) = H^0(X; Z) \oplus G(X).$$

This has a natural structure of a commutative ring (as in [5]) and the augmented total Stiefel-Whitney class can be viewed as a ring homomorphism $KO(X) \xrightarrow{w} \hat{G}(X)$. Recall that the functor $KO(X)$ has a natural transfer homomorphism for finite coverings, [1 and 6]. Our theorem can be stated as follows.

THEOREM 1.1. The functor $\hat{G}(X)$ admits a natural 'transfer homomorphism' for 2-coverings such that the total Stiefel-Whitney class commutes with the transfer, i.e. the diagram

$$\begin{array}{ccc}
KO(X) & \xrightarrow{w} & \hat{G}(X) \\
\downarrow & & \downarrow \\
KO(Y) & \xrightarrow{w} & \hat{G}(Y)
\end{array}$$

is commutative, where $p: X \to Y$ is a 2-covering and the vertical maps are the two transfer homomorphisms.

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Since the transfer in $\hat{G}(X)$ will be defined by an explicit formula, considering the covering $BH \rightarrow BG$ (where $H$ has index 2 in $G$) gives the Evens-Kahn formula.

In [9] G. Segal shows that there is a connective cohomology theory $G^*(X)$ with $G^0(X) = G(X)$. He also asks if there is a cohomology theory $\hat{G}^*(X)$ with $\hat{G}^0(X) = \hat{G}(X)$, and such that the total Stiefel-Whitney class extends to a transformation of cohomology theories. In [10] Snaith showed that this is false if we take $G^*(X) = H^*(X; Z) \oplus G^*(X)$, however, this does not mean that there cannot be a different $\hat{G}^*(X)$ related to $G^*(X)$ in some more complex way. Now, it is well known [6] that every generalized cohomology theory admits a transfer homomorphism for finite coverings such that transformations of cohomology theories commute with the tranfer. We can ask: does the transfer in $\hat{G}(X)$, defined here for 2-coverings, extend to a transfer defined for all finite coverings and possessing all the usual properties of a transfer homomorphism (e.g. see [8])? If so, then it follows that $w$ commutes with the transfer, thus providing strong support for the affirmative answer to Segal’s question.

2. Multiplicative transfers. In this section we shall define the ‘transfer homomorphism’ in $\hat{G}(X)$ which appears in the statement of Theorem 1.1. Our definition will be based on the results of [9]—the alternative approach is as in [2]. The two approaches are equivalent by [7].

Let $G(X)$ be as in the Introduction. By [9] there is a cohomology theory $G^*(X)$ with $G^0(X) = G(X)$. By [6] every generalized cohomology theory possesses a transfer homomorphism for finite coverings; in the above case it shall be called the Segal transfer. Explicitly, the Segal transfer is defined as follows. Let $K_q = S^q \otimes (Z/2)$ as in [9]. Then $K = \{K_q\}_{q \geq 0}$ is a graded topological $Z/2$-module. Let $M(q, n)$ denote the space of maps $K_q \times \cdots \times K_q \to K_{nq}$ which are $n$-linear over $Z/2$ and induce the $n$-fold cup product on homotopy groups. Let $M(n)$ denote the space of graded $n$-linear maps $K \times \cdots \times K \to K$ which induce the graded cup product on homotopy groups. It now follows from [9] that $M(q, n)$ and $M(n)$ are contractible. In [7] it is shown that the natural $\Sigma_n$ action on $M(q, n)$ and $M(n)$ is free. Hence either space can be taken as a model for $E\Sigma_n$. We can now define a map $D: G^0(X) \to G^0(M(n) \times \Sigma_n X^n)$ as follows. Let $G = \bigoplus_{i \geq 1} K(Z/2; i)$ represent $G(X)$ (see [6]). Let $f: X \to G$ represent $[f] \in G(X)$. $D$ is defined by the formula $D([f]) = [D(f)]$, where $D(f): E\Sigma_n \times \Sigma_n X^n \to G$ is given by

$$D(f)(\alpha, x_1, \ldots, x_n) = \alpha(f(x_1), \ldots, f(x_n)).$$

For every $n$-covering $X \to Y$ we have a map $\phi: Y \to E\Sigma_n \times \Sigma_n X^n$ unique up to homotopy, called the pretransfer [5]. The Segal transfer is defined by $N = \phi^* D: G(X) \to G(Y)$, where we identify $E\Sigma_n$ with $M(n)$. Since $N$ is the transfer in a generalised cohomology (Segal’s cohomology) it must possess all the usual properties.

There is also another, closely related ‘transfer’, defined for elements of $H^i(X; Z/2)$ and taking values in $H^{in}(Y; Z/2)$ (this ‘transfer’ has no obvious interpretation in terms of generalized cohomology theories).1 In detail, we define a

map, denoted also by \( D \),

\[
D: H^i(X; \mathbb{Z}/2) \to H^{ni}(E\Sigma_n \times \Sigma_n X^n; \mathbb{Z}/2),
\]

where this time we identify \( E\Sigma_n \) with \( M(i, n) \) by the same formula as above with the obvious reinterpretation of symbols. Just as above, we define \( N: H^i(X; \mathbb{Z}/2) \to H^{ni}(X; \mathbb{Z}/2) \) by \( N = \phi^* D \). It is clear that \( N \) is multiplicative, in the sense that

\[
N(xy) = N(x)N(y) \quad \text{for } x \in H^i(X; \mathbb{Z}/2) \text{ and } y \in H^k(X; \mathbb{Z}/2).
\]

Now let \( X \to Y \) be a 2-covering. We define a ‘transfer’ \( \hat{N}: \hat{G}(X) \to \hat{G}(Y) \) as follows. Let \( \hat{D}: \hat{G}(X) \to \hat{G}(E\Sigma_2 \times \Sigma_2 X^2) \) be given by the formula

\[
\hat{D}(m, 1 + x_1 + \cdots + x_n + \cdots) = \left( 2m, D(1 + x_1 + \cdots + x_n + \cdots) \right) + \sum_{i=1}^{\infty} D(x_i)((1 + t)^{m-i} + 1) + (1 + t)^{m+1},
\]

where \( D \) is defined above and \( t \) is the element of \( H^1(E\Sigma_2 \times \Sigma_2 X^2; \mathbb{Z}/2) \) which is the image of the nonzero element of \( H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \). The augmented transfer for 2-coverings \( \hat{N} \) is now defined by \( \hat{N} = \phi^* \hat{D} \), where \( \phi \) is the pretransfer as above.

We have

**Proposition 2.1.** \( \hat{N} \) is a homomorphism, natural with respect to pull-backs of 2-coverings.

To prove this proposition we need some lemmas. Let \( E\Sigma_2 \times X \to E\Sigma_2 \times X^2 \) be given by \((e, x) \mapsto (e, x, x)\). This is a \( \Sigma_2 \)-equivariant map, and hence induces a map \( \Delta: B\Sigma_2 \times X \to E\Sigma_2 \times \Sigma_2 X^2 \). Let \( i: X^2 \to E\Sigma_2 \times \Sigma_2 X^2 \) be the inclusion \((x, x) \mapsto [(e_0, x, x)]\), where \( e_0 \) is some chosen base point in \( E\Sigma_2 \).

**Lemma 2.2.**

\[
\Delta^* \oplus i^*: H^*(E\Sigma_2 \times \Sigma_2 X^2; \mathbb{Z}/2) \to H^*(B\Sigma_2 \times X; \mathbb{Z}/2) \oplus H^*(X^2; \mathbb{Z}/2)
\]

is injective.

**Proof.** See [11].

**Lemma 2.3.** For the covering \( X \simeq E\Sigma_2 \times X \to B\Sigma_2 \times X \) the pretransfer is given by (the homotopy class of) \( \Delta \).

**Proof.** Easy. For details see [7].

**Lemma 2.4.** For the covering \( X \simeq E\Sigma_2 \times X \to B\Sigma_2 \times X \) the multiplicative transfers defined above are given by

\[
\begin{align*}
(i) \quad N(1 + x_1 + \cdots + x_k \cdots) &= 1 + \sum_{i,k} \text{Sq}^i(x_k)t^{k-i}, \\
(ii) \quad N(x_k) &= \sum_i \text{Sq}^i(x_k)t^{k-i}, \\
(iii) \quad \hat{N}(m, 1 + x_1 + \cdots) &= (2m, (1 + t)^m + \sum_i N(x_i)(1 + t)^{m-i}).
\end{align*}
\]

**Proof.** See [7].

**Proof of Proposition 2.1.** It is clear that \( \hat{N} \) is natural for pull-backs of 2-coverings. From Lemma 2.4(iii) we can easily check that \( \hat{N} \) is a homomorphism for the covering \( X \to B\Sigma_2 \times X \). To prove that \( \hat{N} \) is a homomorphism for all 2-coverings
it is enough to show that \( \hat{D}: \hat{G}(X) \to \hat{G}(E \Sigma_2 \times \Sigma_2, X^2) \) is a homomorphism. By Lemma 2.2 it suffices to prove that \( \Delta^* \hat{D}: \hat{G}(X) \to \hat{G}(B \Sigma_2 \times X) \) and \( \iota^* \hat{D}: \hat{G}(X) \to \hat{G}(X^2) \) are homomorphisms. However, the former follows from Lemma 2.3 and the above, and the latter is obvious from the definition of \( \hat{D} \).

Now let \( F^*(X) \) be any generalized cohomology theory. If \( X \to Y \) is a finite covering, let \( \text{tr}_F: F^0(X) \to F^0(Y) \) denote the Kahn-Priddy transfer.

**Proposition 2.5.** Let \( g: F^0(X) \to \hat{G}(X) \) be a natural homomorphism. Then \( g \) commutes with the transfer for all 2-coverings i.e. \( g(\text{tr}_F(x)) = \tilde{N}(g(x)) \) if and only if \( g \) commutes with the transfer for the coverings \( X \to B \Sigma_2 \times X \).

**Proof.** From the definition of the transfer in a generalized cohomology theory in [6] and from the definition of \( \tilde{N} \) it follows that we only need to show that the diagram

\[
\begin{array}{ccc}
F^0(X) & \xrightarrow{D_F} & F^0(E \Sigma_2 \times \Sigma_2, X^2) \\
\downarrow g & & \downarrow g \\
G(X) & \xrightarrow{\hat{D}} & G(E \Sigma_2 \times \Sigma_2, X^2)
\end{array}
\]

commutes, where \( D_F \) is induced by the Dyer-Lashof map for \( F \) and \( \hat{D} \) is as above. However, this is clear from Lemmas 2.2 and 2.3.

3. **Proof of Theorem 1.1.** Recall that the augmented total Stiefel-Whitney class is a natural ring homomorphism \( w: KO(X) \to \hat{G}(X) \), which takes the real vector bundle \( E \) to \( (\dim E, 1 + w_1(E) + w_2(E) + \cdots) \). By Proposition 2.5 it suffices to prove that the total Stiefel-Whitney class commutes with the transfer for coverings of the form \( X \to B \Sigma_2 \times X \), for all spaces \( X \). Using the naturality of \( \tilde{N} \) and the fact that it is a homomorphism (Proposition 2.1) we easily see that the splitting principle in real \( K \)-theory implies that we only need to show that \( w \text{tr}_{KO}(x) = \tilde{N}w(x) \) for a line bundle \( x \) over \( X \). Suppose \( x \) has the first Stiefel-Whitney class \( w_1(x) \). Then \( \text{tr}_{KO}(x) \in KO(B \Sigma_2 \times X) \) is the sum of two line bundles with first Stiefel-Whitney classes \( w_1(x) \) and \( w_1(x) + t \), where as above \( t \) denotes the image of the nonzero element of \( H^1(B \Sigma_2; Z/2) \). Hence we have

\[
w \text{tr}_{KO}(x) = (2, 1 + w_1(x))(1 + w_1(x) + t) = (2, 1 + t + w_1(x)^2 + tw_1(x)).
\]

On the other hand, from Lemma 2.4 we obtain

\[
\tilde{N}w(x) = \tilde{N}(1, 1 + w_1(x)) = (2, 1 + \text{Sq}^0(w_1(x))t + \text{Sq}^1(w_1(x)) + t) = (2, 1 + w_1(x)t + w_1(x)^2 + t).
\]

This completes the proof.

**References**


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