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THE EVENS-KAHN FORMULA FOR THE TOTAL STIEFEL-WHITNEY CLASS

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ABSTRACT. Let $G(X)$ denote the (augmented) multiplicative group of classical cohomology ring of a space X, with coefficients in $Z/2$. The (augmented) total Stiefel-Whitney class is a natural homomorphism $w: KO(X) \to G(X)$. We show that the functor $G(\ )$ possesses a 'transfer homomorphism' for double coverings such that w commutes with the transfer. This is related to a question of G. Segal. As a special case, we obtain a formula for the total Stiefel-Whitney class of a representation of a finite group induced from a (real) representation of a subgroup of index 2, which is analogous to the one obtained by Evens and Kahn for the total Chern class.

1. Introduction. In [**4**] L. Evens and D. Kahn obtained an interesting formula for the total Chern class of the representation of a finite group induced from a (complex) representation of a subgroup of prime index p (generalizing an earlier formula due to Evens [**3**]). The purpose of this note is to prove an analogous formula for the total Stiefel-Whitney class of the representation of a finite group induced from a (real) representation of a subgroup of index two. In fact, we formulate our statement in the contex of homotopy theory, thus obtaining an interesting reformulation of a question of G. B. Segal [**9**].

Let X be a CW-complex and let $G(X)$ denote the group of units of the commutative ring $\prod_{i\geq 0} H^i(X; Z/2)$. Let

$$\hat{G}(X) = H^0(X; Z) \oplus G(X).$$

This has a natural structure of a commutative ring (as in [**5**]) and the augmented total Stiefel-Whitney class can be viewed as a ring homomorphism $KO(X) \xrightarrow{w} \hat{G}(X)$. Recall that the functor $KO(X)$ has a natural transfer homomorphism for finite coverings, [**1** and **6**]. Our theorem can be stated as follows.

THEOREM 1.1. *The functor $\hat{G}(X)$ admits a natural 'transfer homomorphism' for 2-coverings such that the total Stiefel-Whitney class commutes with the transfer, i.e. the diagram*

$$\begin{array}{ccc} KO(X) & \xrightarrow{w} & \hat{G}(X) \\ \downarrow & & \downarrow \\ KO(Y) & \xrightarrow{w} & \hat{G}(Y) \end{array}$$

is commutative, where $p: X \to Y$ is a 2-covering and the vertical maps are the two transfer homomorphisms.

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Since the transfer in $\hat{G}(X)$ will be defined by an explicit formula, considering the covering $BH \to BG$ (where H has index 2 in G) gives the Evens-Kahn formula.

In [**9**] G. Segal shows that there is a connective cohomology theory $G^*(X)$ with $G^0(X) = G(X)$. He also asks if there is a cohomology theory $\hat{G}^*(X)$ with $\hat{G}^0(X) = \hat{G}(X)$, and such that the total Stiefel-Whitney class extends to a transformation of cohomology theories. In [**10**] Snaith showed that this is false if we take $\hat{G}^*(X) = H^*(X;Z) \oplus G^*(X)$, however, this does not mean that there cannot be a different $\hat{G}^*(X)$ related to $G^*(X)$ in some more complex way. Now, it is well known [**6**] that every generalized cohomology theory admits a transfer homomorphism for finite coverings such that transformations of cohomology theories commute with the tranfer. We can ask: does the transfer in $\hat{G}(X)$, defined here for 2-coverings, extend to a transfer defined for all finite coverings and possessing all the usual properties of a transfer homomorphism (e.g. see [**8**])? If so, then it follows that w commutes with the transfer, thus providing strong support for the affirmative answer to Segal's question.

2. Multiplicative transfers. In this section we shall define the 'transfer homomorphism' in $\hat{G}(X)$ which appears in the statement of Theorem 1.1. Our definition will be based on the results of [**9**]—the alternative approach is as in [**2**]. The two approaches are equivalent by [**7**].

Let $G(X)$ be as in the Introduction. By [**9**] there is a cohomology theory $G^*(X)$ with $G^0(X) = G(X)$. By [**6**] every generalized cohomology theory possesses a transfer homomorphism for finite coverings; in the above case it shall be called the Segal transfer. Explicitly, the Segal transfer is defined as follows. Let $K_q = S^q \otimes (Z/2)$ as in [**9**]. Then $K = \{K_q\}_{q\geq 0}$ is a graded topological $Z/2$-module. Let $M(q,n)$ denote the space of maps $K_q \times \cdots \times K_q \to K_{nq}$ which are n-linear over $Z/2$ and induce the n-fold cup product on homotopy groups. Let $M(n)$ denote the space of graded n-linear maps $K \times \cdots \times K \to K$ which induce the graded cup product on homotopy groups. It now follows from [**9**] that $M(q,n)$ and $M(n)$ are contractible. In [**7**] it is shown that the natural Σ_n action on $M(q,n)$ and $M(n)$ is free. Hence either space can be taken as a model for $E\Sigma_n$. We can now define a map $D\colon G^0(X) \to G^0(M(n)\times_{\Sigma_n} X^n)$ as follows. Let $G = \bigoplus_{i\geq 1} K(Z/2;i)$ represent $G(X)$ (see [**6**]). Let $f\colon X \to G$ represent $[f] \in G(X)$. D is defined by the formula $D([f]) = [D(f)]$, where $D(f)\colon E\Sigma_n \times_{\Sigma_n} X^n \to G$ is given by

$$D(f)(\alpha, x_1, \ldots, x_n) = \alpha(f(x_1), \ldots, f(x_n)).$$

For every n-covering $X \to Y$ we have a map $\phi\colon Y \to E\Sigma_n \times_{\Sigma_n} X^n$ unique up to homotopy, called the pretransfer [**5**]. The Segal transfer is defined by $N = \phi^* D\colon G(X) \to G(Y)$, where we identify $E\Sigma_n$ with $M(n)$. Since N is the transfer in a generalised cohomology (Segal's cohomology) it must possess all the usual properties.

There is also another, closely related 'transfer', defined for elements of $H^i(X;Z/2)$ and taking values in $H^{ni}(Y;Z/2)$ (this 'transfer' has no obvious interpretation in terms of generalized cohomology theories).[1] In detail, we define a

[1]However, see J. P. May, H_∞ *ring spectra and their applications,* Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, p. 242

map, denoted also by D,

$$D\colon H^i(X;Z/2) \to H^{ni}(E\Sigma_n \times_{\Sigma_n} X^n; Z/2),$$

where this time we identify $E\Sigma_n$ with $M(i,n)$ by the same formula as above with the obvious reinterpretation of symbols. Just as above, we define $N\colon H^i(X;Z/2) \to H^{ni}(Y;Z/2)$ by $N = \phi^* D$. It is clear that N is multiplicative, in the sense that $N(xy) = N(x)N(y)$ for $x \in H^i(X;Z/2)$ and $y \in H^k(X;Z/2)$.

Now let $X \to Y$ be a 2-covering. We define a 'transfer' $\hat{N}\colon \hat{G}(X) \to \hat{G}(Y)$ as follows. Let $\hat{D}\colon G(X) \to G(E\Sigma_2 \times_{\Sigma_2} X^2)$ be given by the formula

$$\hat{D}(m, 1 + x_1 + \cdots + x_n + \cdots) = \left(2m, D(1 + x_1 + \cdots + x_n + \cdots) \right.$$
$$\left. + \sum_{i=1}^{\infty} D(x_i)((1+t)^{m-i} + 1) + (1+t)^m + 1\right),$$

where D is defined above and t is the element of $H^1(E\Sigma_2 \times_{\Sigma_2} X^2; Z/2)$ which is the image of the nonzero element of $H^1(BZ/2;Z/2)$. The augmented transfer for 2-coverings $\hat{N}$ is now defined by $\hat{N} = \phi^* D$, where ϕ is the pretransfer as above.

We have

PROPOSITION 2.1. *$\hat{N}$ is a homomorphism, natural with respect to pull-backs of 2-coverings.*

To prove this proposition we need some lemmas. Let $E\Sigma_2 \times X \to E\Sigma_2 \times X^2$ be given by $(e,x) \mapsto (e,x,x)$. This is a Σ_2-equivariant map, and hence induces a map $\Delta\colon B\Sigma_2 \times X \to E\Sigma_2 \times_{\Sigma_2} X^2$. Let $i\colon X^2 \to E\Sigma_2 \times_{\Sigma_2} X^2$ be the inclusion $(x,x) \mapsto [(e_0,x,x)]$, where e_0 is some chosen base point in $E\Sigma_2$.

LEMMA 2.2.

$$\Delta^* \oplus i^*\colon H^*(E\Sigma_2 \times_{\Sigma_2} X^2; Z/2) \to H^*(B\Sigma_2 \times X; Z/2) \oplus H^*(X^2;Z/2)$$

is injective.

PROOF. See [**11**].

LEMMA 2.3. *For the covering $X \simeq E\Sigma_2 \times X \to B\Sigma_2 \times X$ the pretransfer is given by (the homotopy class of) Δ.*

PROOF. Easy. For details see [**7**].

LEMMA 2.4. *For the covering $X \simeq E\Sigma_2 \times X \to B\Sigma_2 \times X$ the multiplicative transfers defined above are given by*

(i) $N(1 + x_1 + \cdots + x_k \cdots) = 1 + \sum_{i,k} \mathrm{Sq}^i(x_k)t^{k-i}$,

(ii) $N(x_k) = \sum_i \mathrm{Sq}^i(x_k)t^{k-i}$,

(iii) $\hat{N}(m, 1 + x_1 + \cdots) = (2m, (1+t)^m + \sum_i N(x_i)(1+t)^{m-i})$.

PROOF. See [**7**].

PROOF OF PROPOSITION 2.1. It is clear that $\hat{N}$ is natural for pull-backs of 2-coverings. From Lemma 2.4(iii) we can easily check that $\hat{N}$ is a homomorphism for the covering $X \to B\Sigma_2 \times X$. To prove that $\hat{N}$ is a homomorphism for all 2-coverings it is enough to show that $\hat{D}: \hat{G}(X) \to \hat{G}(E\Sigma_2 \times_{\Sigma_2} X^2)$ is a homomorphism. By Lemma 2.2 it suffices to prove that $\Delta^* \hat{D}: \hat{G}(X) \to \hat{G}(B\Sigma_2 \times X)$ and $i^* \hat{D}: \hat{G}(X) \to \hat{G}(X^2)$ are homomorphisms. However, the former follows from Lemma 2.3 and the above, and the latter is obvious from the definition of $\hat{D}$.

Now let $F^*(X)$ be any generalized cohomology theory. If $X \to Y$ is a finite covering, let $\mathrm{tr}_F: F^0(X) \to F^0(Y)$ denote the Kahn-Priddy transfer.

PROPOSITION 2.5. *Let $g: F^0(X) \to \hat{G}(X)$ be a natural homomorphism. Then g commutes with the transfer for all 2-coverings i.e. $g(\mathrm{tr}_F(x)) = \hat{N}(g(x))$ if and only if g commutes with the transfer for the coverings $X \to B\Sigma_2 \times X$.*

PROOF. From the definition of the transfer in a generalized cohomology theory in [**6**] and from the definition of $\hat{N}$ it follows that we only need to show that the diagram

$$\begin{array}{ccc} F^0(X) & \xrightarrow{D_F} & F^0(E\Sigma_2 \times_{\Sigma_2} X^2) \\ g\downarrow & & \downarrow g \\ G(X) & \xrightarrow{\hat{D}} & G(E\Sigma_2 \times_{\Sigma_2} X^2) \end{array}$$

commutes, where D_F is induced by the Dyer-Lashof map for F and $\hat{D}$ is as above. However, this is clear from Lemmas 2.2 and 2.3.

3. Proof of Theorem 1.1. Recall that the augmented total Stiefel-Whitney class is a natural ring homomorphism $w: KO(X) \to \hat{G}(X)$, which takes the real vector bundle E to $(\dim E, 1+w_1(E)+w_2(E)+\cdots)$. By Proposition 2.5 it suffices to prove that the total Stiefel-Whitney class commutes with the transfer for coverings of the form $X \to B\Sigma_2 \times X$, for all spaces X. Using the naturality of $\hat{N}$ and the fact that it is a homomorphism (Proposition 2.1) we easily see that the splitting principle in real K-theory implies that we only need to show that $w\,\mathrm{tr}_{KO}(x) = \hat{N}w(x)$ for a line bundle x over X. Suppose x has the first Stiefel-Whitney class $w_1(x)$. Then $\mathrm{tr}_{KO}(x) \in KO(B\Sigma_2 \times X)$ is the sum of two line bundles with first Stiefel-Whitney classes $w_1(x)$ and $w_1(x) + t$, where as above t denotes the image of the nonzero element of $H^1(B\Sigma_2; Z/2)$. Hence we have

$$\begin{aligned} w\,\mathrm{tr}_{KO}(x) &= (2, (1 + w_1(x))(1 + w_1(x) + t)) \\ &= (2, 1 + t + w_1(x)^2 + tw_1(x)). \end{aligned}$$

On the other hand, from Lemma 2.4 we obtain

$$\begin{aligned} \hat{N}w(x) = \hat{N}(1, 1 + w_1(x)) &= (2, 1 + \mathrm{Sq}^0(w_1(x))t + \mathrm{Sq}^1(w_1(x)) + t) \\ &= (2, 1 + w_1(x)t + w_1(x)^2 + t). \end{aligned}$$

This completes the proof.

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