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3-MANIFOLDS WHICH CONTAIN NONPARALLEL PROJECTIVE PLANES

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ABSTRACT. We show that if a closed, connected 3-manifold with a Heegaard splitting of genus three contains mutually disjoint and nonparallel 2-sided projective planes, then the manifold is homeomorphic to the connected sum of $P^2 \times S^1$ and the twisted 2-sphere bundle over the circle.

1. Introduction. In [**7**] Ochiai has shown that if a 3-manifold with a Heegaard splitting of genus two contains a 2-sided projective plane then it is homeomorphic to $P^2 \times S^1$. Negami [**5**] has shown that there exist infinitely many 3-manifolds, each of which has a Heegaard splitting of genus three and contains a 2-sided projective plane. In this paper we show that a 3-manifold with a Heegaard splitting of genus three and two nonparallel 2-sided projective planes is unique.

THEOREM 2. *Let M be a closed, connected 3-manifold with a Heegaard splitting of genus three. Assume M contains two mutually disjoint and nonparallel 2-sided projective planes. Then M is homeomorphic to the connected sum $P^2 \times S^1 \# K$, where K denotes the twisted 2-sphere bundle over the circle.*

We note that there are infinitely many irreducible, closed 3-manifolds, each of which contains mutually disjoint and nonparallel 2-sided projective planes (Negami [**6**], Row [**9**]).

Let M be a closed, connected, prime 3-manifold. Then as consequences of Theorem 2 we have

COROLLARY 1. *If M contains two mutually disjoint and nonparallel 2-sided projective planes, then the Heegaard genus of M is greater than three.*

COROLLARY 2. *If the minimal number of generators of $\pi_2(M)$ is greater than one, then the Heegaard genus of M is greater than three.*

We work throughout in the piecewise linear category. For definitions of Heegaard splitting and other standard terms in three-dimensional topology, we refer to [**2**].

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2. Theorem 1. In this section, for the proof of Theorem 2, we will slightly modify Ochiai's result [**8**] as Theorem 1.

THEOREM 1. *Let M be a closed, connected, prime 3-manifold with a Heegaard splitting (V_1, V_2). Assume M contains 2-sided projective planes $P_1, \dots, P_n$ $(n \geq 2)$*

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which are mutually disjoint and nonparallel. Then there is an ambient isotopy h_t $(0 \leq t \leq 1)$ *of* M *such that* $h_1(P_1) \cap V_1, \ldots, h_1(P_n) \cap V_1$ *are mutually nonparallel meridian disks of* V_1.

LEMMA 2.1. *Let* $M, (V_1, V_2), P_1, \ldots, P_n$ *be as in Theorem 1. Then there is an ambient isotopy* g_t $(0 \leq t \leq 1)$ *of* M *such that* $g_1(P_i) \cap V_1$ $(i = 1, \ldots, n)$ *is a disk.*

PROOF. This can be proved by the argument of inverse operations of an isotopy of type A [**3**] defined in [**8**]. In [**8**] Ochiai considered one projective plane but this argument applies to finitely many mutually disjoint 2-sided projective planes.

LEMMA 2.2. *Let* H *be a solid Klein bottle and* Q *a 2-sided Möbius band properly embedded in* H. *If we attach a 2-handle to* H *along* ∂Q *then we get* $P^2 \times I$.

PROOF. Let D be a meridian disk of H, i.e. D cuts H into a 3-cell. By [**4**] we may suppose ∂D intersects ∂Q transversely in two points. So we may suppose $Q \cap D$ consists of an arc and some simple loops. Since H is irreducible we can move D by an isotopy so that $Q \cap D$ consists of an arc. Then H is homeomorphic to $Q \times I$, where Q corresponds to $Q \times \{1/2\}$, and we have the conclusion of Lemma 2.2.

PROOF OF THEOREM 1. By Lemma 2.1 there is an ambient isotopy g_t of M such that $D_i = g_1(P_i) \cap V_1$ $(i = 1, \ldots, n)$ is a disk. If D_i separates V_1 then P_i separates M into M_1 and M_2. Let $D(M_1)$ be a double of M_1. By Poincaré duality we see that $\chi(D(M_1)) = 0$. On the other hand, we easily see that $\chi(D(M_1)) = 2\chi(M_1) - \chi(\partial M_1) = 2\chi(M_1) - 1$. Hence $\chi(M_1) = 1/2$, a contradiction.

Assume that some D_i and D_j, say D_1 and D_2, are parallel in V_1. There is an annulus A in $F = \partial V_1$ such that $A \cap (D_1 \cup D_2) = \partial A = \partial D_1 \cup \partial D_2$. Let $E = A \cup (V_2 \cap P_1) \cup (V_2 \cap P_2)$. E is a 2-sided Klein bottle in V_2. By the loop theorem [**2**] and irreducibility of V_2, we see that E bounds a solid Klein bottle H in V_2. On the other hand, $A \cup D_1 \cup D_2$ bounds a 3-cell C in V_1. By Lemma 2.2 we get $P^2 \times I$ by attaching C to H along A, which contradicts the fact that P_1 and P_2 are not parallel.

This completes the proof of Theorem 1.

3. Theorem 2. Let $S^1 = \{z \in \mathbf{C}\colon |z| = 1\}$ and $D^2 = \{z \in \mathbf{C}\colon |z| \leq 1\}$. Then we get $P^2 \times S^1$ from $D^2 \times S^1$ by identifying its boundary points in the following manner:

$$(z_1, z_2) \sim (-z_1, z_2) \qquad (z_1 \in \partial D^2).$$

Let $l_1 = \{0\} \times S^1$ be a simple loop in $P^2 \times S^1$, and l_2 an essential, simple loop in $P_1^2 = D^2 \times \{1\}/\sim$ which intersects l_1 in a single point. We note that l_2 is unique up to an isotopy of P_1^2. Let $K_1 = \mathrm{cl}(P^2 \times S^1 - N(l_2))$, $V_1 = N(l_1 \cup l_2)$, $V_2 = \mathrm{cl}(P^2 \times S^1 - V_1)$, where $N(X)$ denotes a regular neighborhood of a polyhedron X. Then we easily see that (V_1, V_2) is a genus two Heegaard splitting of $P^2 \times S^1$.

LEMMA 3.1. *Let* P_1, P_2 *be the components of* $\partial(P^2 \times I)$ *and* D_i $(i = 1, 2)$ *a disk in* P_i. *If* M *is obtained from* $P^2 \times I$ *by identifying* D_1 *and* D_2, *then* M *is homeomorphic to* K_1.

PROOF. We give D_1 and D_2 fixed orientations. Then we have two possibilities when we attach D_1 to D_2 depending on whether the two orientations coincide or differ. But if we slide D_1 along an orientation reversing loop in P_1, we get the same manifolds by each of the above attachings.

Let $D = P_1^2 - N(l_2)$. Then D is a disk properly embedded in K_1 and D cuts K_1 into $P^2 \times I$. Hence M is homeomorphic to K_1.

LEMMA 3.2. *Let V be a genus two handlebody and Q a nonseparating 2-sided Möbius band properly embedded in V. If M is obtained from V by attaching a 2-handle along ∂Q, then M is homeomorphic to K_1.*

PROOF. We easily find a disk D in V such that $D \cap \partial V = \partial D \cap \partial V = \alpha$ is an arc, $D \cap Q = \partial D \cap Q = \beta$ is an essential arc of Q and $\alpha \cup \beta = \partial D$, $\alpha \cap \beta = \partial\alpha = \partial\beta$. By performing surgery on Q along D we get a disk D' properly embedded in V. Since Q is nonseparating in V, D' is a meridian disk. Since Q is 2-sided, we can move D' by a small isotopy so that $D' \cap Q = \emptyset$. Let V' be V cut along D'. V' is a solid Klein bottle, for it contains a 2-sided Möbius band Q. By Lemma 2.2 we get $P^2 \times I$ by attaching a 2-handle to V' along ∂Q. Let D_1, D_2 be the copies of D' on $\partial V'$. Since Q is nonseparating in V, D_1 and D_2 are contained in mutually distinct components of $\partial(P^2 \times I)$. Hence by Lemma 3.1, M is homeomorphic to K_1.

PROPOSITION. *Let M be a closed, connected 3-manifold which has a Heegaard splitting (V_1, V_2) of genus three. Assume there are mutually disjoint and nonparallel 2-sided projective planes P_1, P_2 in M such that $D_1 = P_1 \cap V_1$ (resp. $D_2 = P_2 \cap V_2$) is a disk and $Q_1 = P_1 \cap V_2$ (resp. $Q_2 = P_2 \cap V_1$) is a Möbius band. Then M is homeomorphic to $P^2 \times S^1 \# K$, where K is the twisted 2-sphere bundle over the circle.*

PROOF. Since P_i $(i = 1, 2)$ is nonseparating in M, D_1 (resp. D_2) is a meridian disk of V_1 (resp. V_2).

Then we claim that $D_1 \cup Q_2$ (resp. $D_2 \cup Q_1$) does not separate V_1 (resp. V_2). Assume $D_1 \cup Q_2$ separates V_1. If $D_2 \cup Q_1$ does not separate V_2, we can find a loop l in ∂V_2 such that l intersects $\partial D_2 \cup \partial Q_1$ transversely in a single point, which contradicts the fact that $D_1 \cup Q_2$ separates V_1. So $D_2 \cup Q_1$ also separates V_2. Let V' be V_1 cut along D_1, and D_1', D_1'' the copies of D_1 in $\partial V'$. Then V' is a genus two handlebody which contains a 2-sided Möbius band Q_2. Then we have two cases.

Case 1. Q_2 is parallel to a Möbius band Q' in $\partial V'$.

In this case D_1' or D_1'' is contained in Q', for if not then P_2 is isotopic into V_2, a contradiction. Moreover, since Q_2 is nonseparating in V_1, Q' contains only one of D_1' and D_1''. Hence $P_2 \cup D_1$ cuts V_1 into a solid Klein bottle and the other component.

Case 2. Q_2 is not boundary parallel.

In this case by the argument of the proof of Lemma 1 of [**7**], we see that there is a complete system of meridian disks $\{D', D''\}$ of V' such that $D' \cap Q_2 = \emptyset$ and $D'' \cap Q_2$ is an essential arc of Q_2. We may suppose

$$(D' \cup D'') \cap (D_1' \cup D_1'') = \emptyset.$$

D' cuts V' into a solid Klein bottle V'' for it contains a 2-sided Möbius band Q_2. Then, by the proof of Lemma 2.2, Q_2 cuts V'' into two solid Klein bottles. Since $D_1 \cup Q_2$ separates V_1, the two copies of D_1 are on mutually distinct components of

V'' cut along Q_2, and the two copies of D' are on the same component of V'' cut along Q_2. Hence $D_1 \cup Q_2$ cuts V_1 into a solid Klein bottle and the other component.

So in either case $D_1 \cup Q_2$ cuts V_1 into a solid Klein bottle R_1 and the other component. By the same argument $D_2 \cup Q_1$ cuts V_2 into a solid Klein bottle R_2 and the other component. Let D_2', Q_1' (resp. D_1', Q_2') be the copies of D_2, Q_1 (resp. D_1, Q_2) on ∂R_1 (resp. ∂R_2). By considering the Euler characteristic we see that $\mathrm{cl}(\partial R_1 - (D_2' \cup Q_1'))$ and $\mathrm{cl}(\partial R_2 - (D_1' \cup Q_2'))$ are identified in M. By Lemma 2.2 we get $P^2 \times I$ by attaching $N(D_2')$ (resp. $N(D_1')$) to R_2 (resp. R_1) and so we get $P^2 \times I$ by attaching R_1 to R_2 along $\mathrm{cl}(\partial R_1 - (D_2' \cup Q_1'))$ and $\mathrm{cl}(\partial R_2 - (D_1' \cup Q_2'))$. But this contradicts the fact that P_1 and P_2 are not parallel.

Hence $D_1 \cup Q_2$ (resp. $D_2 \cup Q_1$) does not separate V_1 (resp. V_2) and the claim is established.

Let V_1' (resp. V_2') be V_1 cut along D_1 (resp. V_2 cut along D_2) and K_1' (resp. K_1'') be the manifold obtained by attaching $N(D_1)$ (resp. $N(D_2)$) to V_2' (resp. V_1'). By Lemma 3.2 and the above claim we see that each of K_1' and K_1'' is homeomorphic to K_1. Let D be a disk in K_1, defined in the proof of Lemma 3.2, and D' (resp. D'') the corresponding disk in K_1' (resp. K_1''). Then by [**4**], $\partial D'$ and $\partial D''$ are isotopic in $K_1' \cap K_1'' = \partial K_1' = \partial K_1''$. So we may suppose $\partial D'$ and $\partial D''$ are identified in M and then $D' \cup D''$ is a nonseparating 2-sphere in M. So by Lemmas 3.8 and 3.17 of [**2**], $M = M_1 \# K$. By Corollary II.10 of [**3**], M_1 has a Heegaard splitting of genus two, and by using a cut and paste method on P_1 we see that M_1 contains a 2-sided projective plane. Hence by [**7**], M_1 is $P^2 \times S^1$ and this completes the proof of the Proposition.

PROOF OF THEOREM 2. Let $(V_1,\ V_2)$ be a Heegaard splitting of genus three of M. If M is prime then by Theorem 1 we may suppose $P_i \cap V_1$ $(i = 1,2)$ is a disk. Then by performing an isotopy of type A [**3**] on P_i, $i = 1$ or 2, say 2, we can move P_1, P_2 to the position as in the Proposition. If M is not prime, then by Corollary II.10 of [**3**] and [**7**], M is either $P^2 \times S^1 \# K$ or $P^2 \times S^1 \# L_n$, where L_n denotes a three-dimensional lens space. We note that the orientable three manifold with a Heegaard splitting of genus one is either a lens space or $S^2 \times S^1$, and the nonorientable 3-manifold with a Heegaard splitting of genus one is K [**2**, **4**]. Assume M is $P^2 \times S^1 \# L_n$. Then there is a 2-sphere S in M such that S cuts M into $P^2 \times S^1 - \mathrm{Int}\, B_1^3$ and $L_n - \mathrm{Int}\, B_2^3$, where B_1^3 (resp. B_2^3) is a 3-cell in $P^2 \times S^1$ (resp. L_n). Since $P^2 \times S^1$ and L_n are irreducible, there is an ambient isotopy g_t $(0 \leq t \leq 1)$ of M such that

$$g_1(P_i) \subset P^2 \times S^1 - \mathrm{Int}\, B_1^3 \qquad (i = 1,2).$$

Hence $g_1(P_1)$ and $g_2(P_2)$ are parallel in M, a contradiction. So M is either $P^2 \times S^1 \# K$ or $P^2 \times S^1 \# S^2 \times S^1$. But by Lemma 3.17 of [**2**] these are pairwise homeomorphic. This completes the proof of Theorem 2.

PROOF OF COROLLARIES. Corollary 1 is an immediate consequence of Theorem 2. So we will prove Corollary 2. Suppose the minimal number of generators of $\pi_2(M)$ is n (≥ 2). By the projective plane theorem [**1**] there is a system of mutually disjoint 2-spheres and 2-sided projective planes $\{Q_1, \ldots, Q_n\}$ which represents a generator of $\pi_2(M)$. Assume that some Q_i is a 2-sphere. Since M is prime, by Lemma 3.13 of [**2**] we see that M is a 2-sphere bundle over a circle. But this contradicts that $n \geq 2$. Hence each Q_i is a projective plane. Clearly $\{Q_1, \ldots, Q_n\}$

are mutually nonparallel. So by Corollary 1 the Heegaard genus of M is greater than three.

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