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A NONLOCALLY CONNECTED CONTINUUM X SUCH THAT $C(X)$ IS A RETRACT OF 2^X

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ABSTRACT. Let X be a metric continuum and let 2^X ($C(X)$) denote the hyperspace of closed subsets (subcontinua) of X. An example is given of a nonlocally connected continuum X such that $C(X)$ is a retract of 2^X.

By a *continuum* we mean a compact connected metric space. If X is a continuum, then 2^X (respectively, $C(X)$) denotes the hyperspace of closed subsets (respectively, subcontinua) of X, each with the Hausdorff metric. In this note we will obtain a partial answer to a question raised by Sam B. Nadler, Jr. [**8**] by giving an example of a nonlocally connected continuum X such that $C(X)$ is a retract of 2^X.

In 1939 Wojdyslawski [**10**] proved that $C(X)$ is an absolute retract if and only if X is locally connected. Hence, $C(X)$ is a retract of 2^X when X is locally connected. In Theorem 3.6 of [**8**], Nadler shows that $C(X)$ is always a continuous image of 2^X. Following this result, in Question 3.7 of [**8**], Nadler asks, "When is $C(X)$ a retract of 2^X?" Nadler discusses this question further in [**6**] and in Chapter 6 of [**7**]. In particular, in Theorem 6.11 of [**7**], Nadler proves a result which indicates that, when X is not locally connected, a retraction from 2^X onto $C(X)$, if one exists, must be rather complicated. In [**2**] the author determined the first partial answer to Nadler's question by giving an example of a nonlocally connected continuum X such that $C(X)$ is not a retract of 2^X. In [**5**] Lawson attempted to give some conditions which would imply the existence of a retraction from 2^X onto $C(X)$ for certain nonlocally connected continua. However, as noted in [**2**], Lawson's proof was not correct. We will show that for one of the examples in Lawson's paper, namely the cone over a convergent sequence, there does exist a retraction from 2^X onto $C(X)$. It is interesting to note that this example and the example in [**2**] are both smooth dendroids. Thus, even within this rather well-behaved class of continua, $C(X)$ may or may not be a retract of 2^X when X is not locally connected.

We observe in Corollary 1 that the example in this paper also has the property that $F_1(X)$, the space of singleton subsets of X, is a retract of 2^X. This is the first example of a nonlocally connected continuum for which it has been shown that $F_1(X)$ is a retract of 2^X. Hence, the example also gives a partial answer to Nadler's question in 6.2 of [**7**].

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The symbol I will denote the closed unit interval of real numbers and the symbol $\mathbf{Z}^+$ will denote the set of positive integers. For the remainder of this paper, the symbol X will denote the specific continuum defined in the following paragraph. We will let d denote the usual Euclidean metric on X and ρ denote the Hausdorff metric on 2^X. An order arc β in $C(X)$ is an arc with the property that whenever $A, B \in \beta$, then $A \subset B$ or $B \subset A$.

We describe the planar continuum X in terms of polar coordinates. Let $T = \{0\} \cup \{1/n | n \in \mathbf{Z}^+\}$. For each $\theta \in T$, let $L_\theta = \{(r, \theta) | r \in I\}$. Let $X = \bigcup\{L_\theta | \theta \in T\}$, $p = (0,0)$, $2^X_p = \{A \in 2^X | p \in A\}$, and $\mathscr{A} = 2^X_p \cup (\bigcup\{2^{L_\theta} | \theta \in T\})$. We note that $\mathscr{A}$ is closed and that $C(X) \subset \mathscr{A}$.

We now list all of the functions that we use to define a retraction R from 2^X onto $C(X)$. Define mappings $r_{\min}, r_{\max}\colon 2^X \to I$ by

$$r_{\min}(A) = \min\{r | (r, \theta) \in A\} \quad \text{and} \quad r_{\max}(A) = \max\{r | (r, \theta) \in A\}.$$

Define mappings $\theta_{\min}, \theta_{\max}\colon 2^X - 2^X_p \to I$ by

$$\theta_{\min}(A) = \min\{\theta | (r, \theta) \in A\} \quad \text{and} \quad \theta_{\max}(A) = \max\{\theta | (r, \theta) \in A\}.$$

Define the mapping $g\colon 2^X - 2^X_p \to C(X)$ by

$$g(A) = \{(r, \theta_{\min}(A)) | r_{\min}(A) \leqslant r \leqslant r_{\max}(A)\}.$$

Define the function $f\colon 2^X \to C(X)$ by

$$f(A) = \bigcap\{M \in C(X) | A \subset M\}.$$

Using Theorem 1 of [**1**] and Theorems 1,2 and 3 of [**4**], it is easy to show that if $A \in 2^X$ and $A \not\subset L_0 - \{p\}$, then f is continuous at A. Define the mapping α: $(2^X - \mathscr{A}) \times I \to C(X)$ by

$$\alpha(A, s) = \begin{cases} h_1(g(A), 3s) & \text{if } s \in [0, \frac{1}{3}], \\ f(\{p\} \cup g(A)) \cup h_2(A, 3s - 1) & \text{if } s \in [\frac{1}{3}, \frac{2}{3}], \\ f(A) \cup h_2(\{(r_{\max}(A), \theta_{\min}(A))\}, 3 - 3s) & \text{if } s \in [\frac{2}{3}, 1], \end{cases}$$

where h_1 and h_2 are as defined on p. 36 of [**3**]. If $A \in 2^X - \mathscr{A}$, then $\{\alpha(A, s) | s \in [0, \frac{1}{3}]\}$ is the unique order arc in $C(X)$ from $g(A)$ to $f(\{p\} \cup g(A))$, $\{\alpha(A, s) | s \in [\frac{1}{3}, \frac{2}{3}]\}$ is the order arc in $C(X)$ which grows uniformly from $f(\{p\} \cup g(A))$ to $f(\{p\} \cup g(A)) \cup f(A)$, and $\{\alpha(A, s) | s \in [\frac{2}{3}, 1]\}$ is the unique order arc (possibly degenerate) in $C(X)$ from $f(A)$ to $f(\{p\} \cup g(A)) \cup f(A)$. Thus, for each $A \in 2^X - \mathscr{A}$, $\{\alpha(A, s) | s \in I\}$ is an arc in $C(X)$ from $g(A)$ to $f(A)$. Define the function R: $2^X \to C(X)$ by

$$(1) \qquad R(A) = \begin{cases} f(A) & \text{if } A \in \mathscr{A}, \\ \alpha(A, \min\{\theta_{\max}(A)/r_{\min}(A), 1\}) & \text{if } A \in 2^X - \mathscr{A}. \end{cases}$$

THEOREM 1. *Let R be as defined in* (1). *Then R is a retraction from 2^X onto $C(X)$.*

PROOF. It is clear that R is continuous on the open set $2^X - \mathscr{A}$. Let $A \in \mathscr{A}$. If $A \in 2^X_p$, then f is continuous at A. If $A \in \mathscr{A} - 2^X_p$, then $f(A) = g(A)$. Since g is continuous at A, it follows that the restriction of f to the closed set $\mathscr{A}$ is continuous at A. Hence, the restriction of R to the closed set $\mathscr{A}$ is continuous. Thus, to complete the proof that R is continuous, it will suffice to show that if $A \in \mathscr{A}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence in $2^X - \mathscr{A}$ such that $A_n \to A$, then $R(A_n) \to R(A)$. Let $\varepsilon > 0$. For $A \in \mathscr{A}$, we will consider three distinct cases.

Case I. Suppose that $p \in A$ and $A \not\subset L_0$. Since f is continuous at A, there exists $\delta > 0$ such that if $B \in 2^X$ and $\rho(A, B) < \delta$, then $\rho(f(A), f(B)) < \varepsilon$. Since $A_n \to A$, there exists $n_1 \in \mathbf{Z}^+$ such that $n \geqslant n_1$ implies $\rho(A_n, A) < \delta/2$. For some $\theta > 0$, $A \cap (L_\theta - \{p\}) \neq \varnothing$. It follows that there exists $n_2 \in \mathbf{Z}^+$ such that $n \geqslant n_2$ implies $\theta_{\max}(A_n) > \theta/2$. Thus, since $r_{\min}(A_n) \to r_{\min}(A) = 0$, there exists $n_3 \in \mathbf{Z}^+$ such that $n \geqslant n_3$ implies $\theta_{\max}(A_n)/r_{\min}(A_n) > 1$. Let $n \in \mathbf{Z}^+$ such that $n \geqslant \max\{n_1, n_3\}$. Then

$$R(A_n) = \alpha(A_n, 1) = f(A_n) \quad \text{and} \quad \rho(R(A_n), R(A)) = \rho(f(A_n), f(A)) < \varepsilon.$$

Case II. Suppose that $p \in A$ and $A \subset L_0$. Let δ and n_1 be as in Case I. Since $r_{\min}(A_n) \to r_{\min}(A) = 0$, there exists $n_4 \in \mathbf{Z}^+$ such that $n \geqslant n_4$ implies $r_{\min}(A_n) < \delta/2$. Let $n \in \mathbf{Z}^+$ such that $n \geqslant \max\{n_1, n_4\}$. If $\theta_{\max}(A_n) \geqslant r_{\min}(A_n)$, then

$$R(A_n) = \alpha(A_n, 1) = f(A_n) \quad \text{and} \quad \rho(R(A_n), R(A)) = \rho(f(A_n), f(A)) < \varepsilon.$$

Now suppose that $\theta_{\max}(A_n) \leqslant r_{\min}(A_n) < \delta/2$. Let $\pi(A_n) = \{(r, \theta_{\min}(A_n)) | (r, \theta) \in A_n\}$. Since $\theta_{\min}(A_n) \leqslant \theta_{\max}(A_n) < \delta/2$, $\rho(\pi(A_n), A_n) < \delta/2$. Thus,

$$\rho(\pi(A_n), A) \leqslant \rho(\pi(A_n), A_n) + \rho(A_n, A) < \delta/2 + \delta/2 = \delta.$$

Hence, $\rho(f(\pi(A_n)), f(A)) = \rho(g(A_n), f(A)) < \varepsilon$. Since $\rho(\pi(A_n), A) < \delta$ and since $p \in A$, $\rho(\{p\} \cup \pi(A_n), A) < \delta$. Hence,

$$\rho(f(\{p\} \cup \pi(A_n)), f(A)) = p(f(\{p\} \cup g(A_n)), f(A)) < \varepsilon.$$

Since $\rho(f(\{p\} \cup g(A_n)), f(A)) < \varepsilon$ and $\rho(f(A_n), f(A)) < \varepsilon$,

$$\rho(f(\{p\} \cup g(A_n)) \cup f(A_n), f(A)) < \varepsilon.$$

Let $s \in I$. Then $g(A_n) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n))$ or $f(\{p\} \cup g(A_n)) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n)) \cup f(A_n)$ or $f(A_n) \subset \alpha(A_n, s) \subset f(\{p\} \cup g(A_n)) \cup f(A_n)$. It follows that for each $s \in I$, $\rho(\alpha(A_n, s), f(A)) < \varepsilon$. Since $R(A_n) \in \{\alpha(A_n, s) | s \in I\}$, we conclude that

$$\rho(R(A_n), R(A)) = \rho(R(A_n), f(A)) < \varepsilon.$$

Case III. Suppose that $p \notin A$. Since $\{A_n\}_{n=1}^{\infty} \subset 2^X - \mathscr{A}$ and since $A_n \to A$, it follows that $A \subset L_0 - \{p\}$. Then $R(A) = f(A) = g(A)$. Since h_1 is uniformly continuous, there exists $\delta_1 > 0$ such that if $B_1, B_2 \in C(X)$ and $s_1, s_2 \in I$ such that $\rho(B_1, B_2) < \delta_1$ and $|s_1 - s_2| < \delta_1$, then $\rho(h_1(B_1, s_1), h_1(B_2, s_2)) < \varepsilon$. Since g is continuous at A, there exists $n_5 \in \mathbf{Z}^+$ such that $n \geqslant n_5$ implies $\rho(g(A_n), g(A)) < \delta_1$. Since $\theta_{\max}(A_n) \to \theta_{\max}(A) = 0$ and $r_{\min}(A_n) \to r_{\min}(A) > 0$, there exists $n_6 \in \mathbf{Z}^+$ such that $n \geqslant n_6$ implies $\theta_{\max}(A_n)/r_{\min}(A_n) < \min\{\delta_1/3, 1/3\}$. Let $n \in \mathbf{Z}^+$ such that $n \geqslant \max\{n_5, n_6\}$. Then

$$\begin{aligned} R(A_n) &= \alpha(A_n, \theta_{\max}(A_n)/r_{\min}(A_n)) \\ &= h_1(g(A_n), 3(\theta_{\max}(A_n)/r_{\min}(A_n))). \end{aligned}$$

Since $\rho(g(A_n), g(A)) < \delta_1$ and

$$|3(\theta_{\max}(A_n)/r_{\min}(A_n)) - 0| < 3(\delta_1/3) = \delta_1$$

and since $h_1(g(A), 0) = g(A)$,

$$\rho(h_1(g(A_n), 3(\theta_{\max}(A_n)/r_{\min}(A_n))), h_1(g(A), 0)) = \rho(R(A_n), R(A)) < \varepsilon.$$

The proof that R is continuous is now complete. Since $C(X) \subset \mathscr{A}$, $R(A) = f(A) = A$ for each $A \in C(X)$. Hence, R is a retraction from 2^X onto $C(X)$.

COROLLARY 1. *Let* $F_1(X) = \{\{x\} | x \in X\}$. *Then* $F_1(X)$ *is a retract of* 2^X.

PROOF. Since X is a smooth dendroid, it follows from Theorem 1 of **[9]** that $F_1(X)$ is a retract of $C(X)$. By Theorem 1, $C(X)$ is a retract of 2^X. Hence, $F_1(X)$ is a retract of 2^X.

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