

SURGERY UP TO HOMOTOPY EQUIVALENCE FOR NONPOSITIVELY CURVED MANIFOLDS

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ABSTRACT. Let M^n be a smooth closed manifold which admits a metric of nonpositive curvature. We show, using a theorem of Farrell and Hsiang, that if $n + k \geq 6$, then the surgery obstruction map $[M \times D^k, \partial; G/\text{TOP}] \rightarrow L_{n+k}^h(\pi_1 M, w_1(M))$ is injective, where L_*^h are the obstruction groups for surgery up to homotopy equivalence.

Farrell and Hsiang have shown that if M^n is a closed nonpositively curved Riemannian manifold and $n + k \geq 6$, then the surgery map

$$[M^n \times D^k, \partial; G/\text{TOP}, *] \xrightarrow{\theta_s} L_{n+k}^s(\pi_1 M, w_1(M))$$

for topological surgery in a split monomorphism [1]. This note considers the implications of their result of h -surgery, i.e. surgery up to homotopy equivalence rather than simple homotopy equivalence.

LEMMA. Let Q^p be a compact manifold such that $\theta_s: [Q^p, \partial; G/\text{TOP}, *] \rightarrow L_p^s(\pi_1 Q, w_1(Q))$ is monic, let N^n be a compact, connected submanifold of Q^p which has a normal microbundle $\nu_Q(N)$, and let $f: U^p \rightarrow Q^p$ be a simple homotopy equivalence which is a homeomorphism near the boundary. Then any normal map $f|_V: V = f^{-1}(N) \rightarrow N$ induced by making f transverse to N is normally cobordant to a homeomorphism.

PROOF. Recall that the long exact sequence of topological surgery

$$\begin{aligned} \rightarrow \mathfrak{S}^a(Q^p \times D^k, \partial) \xrightarrow{\eta_a} [Q^p \times D^k, \partial; G/\text{TOP}, *] \xrightarrow{\theta_a} L_{p+k}^a(\pi_1 Q, w_1(Q)) \\ \xrightarrow{\partial} \mathfrak{S}^a(Q^p \times D^{k-1}, \partial) \xrightarrow{\eta_a} \dots \rightarrow [Q^p, \partial; G/\text{TOP}, *] \xrightarrow{\theta_a} L_p^a(\pi_1 Q, w_1(Q)) \end{aligned}$$

(where $a = s$ or h denotes surgery up to simple homotopy equivalence or up to homotopy equivalence, respectively) is a long exact sequence of groups and group homomorphisms (see [4]; also see [2] for surgery in the topological category). Let $x = [f: U \rightarrow Q] \in \mathfrak{S}^s(Q, \partial)$ and let $i: N \hookrightarrow Q$ be the inclusion map. Then the element $[f|_V: V \rightarrow N] \in [N, \partial; G/\text{TOP}, *]$ defined above is $i^*(\eta_s(x))$, where $i^*: [Q, \partial; G/\text{TOP}, *] \rightarrow [N, \partial; G/\text{TOP}, *]$. Since θ_s is a monomorphism, $\text{image}(\eta_s) = 0$ in $[Q, \partial; G/\text{TOP}, *]$, and since i^* is a homomorphism of groups we see that

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$i^*(\eta_s(x)) = i^*(0) = 0$, or that $i^*(\eta_s(x))$ is normally cobordant to a homeomorphism.

Geometrically, we have made the following construction. $\text{image}(\eta_s) = 0$ implies that each element x of the s -structure set $\mathcal{S}^s(Q, \partial)$ is obtained from a homeomorphism of Q by the action of an element of $L_{p+1}^s(\pi_1 Q, w_1(Q))$. That is, there is a normal map $[(X^{p+1}, \partial_0 X, \partial_1 X) \xrightarrow{g} (Q \times I, Q \times 0, Q \times 1)]$ so that $g|_{\partial_0 X}: \partial_0 X \rightarrow Q \times 0$ is a homeomorphism and $g|_{\partial_1 X}: \partial_1 X \rightarrow Q \times 1$ is a simple homotopy equivalence representing $x \in \mathcal{S}^s(Q, \partial)$. Make g transverse to $N \times I \subset Q \times I$ and let $Y = g^{-1}(N \times I)$. Then $[g^{-1}(N \times 1) \rightarrow N \times 1]$ represents $i^*(\eta_s(x))$ and $g|_Y: Y \rightarrow N \times I$ is the desired normal bordism.

PROPOSITION. *If M^n is a closed, nonpositively curved manifold then $\theta_h: [M \times D^k, \partial; G/\text{TOP}, *] \rightarrow L_{n+k}^h(\pi_1 M, w_1(M))$ is split monic if $n + k \geq 6$.*

PROOF. We may use the Lemma to show that θ_h is monic, i.e. that $\text{image}(\eta_h) = 0$. Let $x \in \mathcal{S}^h(M \times D^k, \partial)$ be represented by a homotopy equivalence $f: (V^{n+k}, \partial) \rightarrow (M \times D^k, \partial)$ which is a homeomorphism near the boundary. Since taking products with the circle kills Whitehead torsion [3], $f \times \text{Id}_{S^1}: V \times S^1 \rightarrow M \times D^k \times S^1$ is a simple homotopy equivalence. Since M is nonpositively curved, so is $M \times S^1$, and [1] shows that θ_s is monic for $M \times S^1$. Apply the Lemma with $Q = M \times S^1$ and $N = M \times 1$ to conclude that f is normally cobordant to a homeomorphism, i.e. $\eta_h(x) = 0$.

The splitting in [1] of θ_s induces a splitting of θ_h if we consider this commutative diagram:

$$\begin{array}{ccc} [M^n, \partial; G/\text{TOP}, *] & \xrightarrow{\theta_h} & L_n^h(\pi_1 M, w_1(M)) \\ p^* \downarrow & & \downarrow \text{tr}(p) \\ [M^n \times S^1, \partial; G/\text{TOP}, *] & \xrightarrow{\theta_s} & L_{n+1}^s(\pi_1 M \times \mathbf{Z}, w_1(M) \circ \text{pr}_1) \end{array}$$

Here $p: M^n \times S^1 \rightarrow M^n$ is a first-coordinate projection and $\text{tr}(p)$ is the transfer associated to p [4, §4.2]. (The effect of $\text{tr}(p)$ on one of the “objects” used in defining $L_n^h(\pi_1 M, w_1(M))$ in [5, Chapters 9 and 17D] is to multiply a map $(B, A) \rightarrow (Y, X)$ inducing a homotopy equivalence $A \rightarrow X$ by Id_{S^1} ; by [3] this gives a map $(B \times S^1, A \times S^1) \rightarrow (Y \times S^1, X \times S^1)$ inducing a simple homotopy equivalence $A \times S^1 \rightarrow X \times S^1$, i.e. an “object” for the definition of $L_{n+1}^s(\pi_1 M \times \mathbf{Z}, w_1(M) \circ \text{pr}_1)$.) If $i: M \rightarrow M \times S^1$ is the inclusion map, then i^* splits p^* . Compose i^* with the splitting of [1] for θ_s and with $\text{tr}(p)$ to split θ_h .

More generally, the Proposition holds for any closed manifold M^n such that M^n and $M^n \times S^1$ both satisfy condition (*) of [1, p. 201].

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