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SURGERY UP TO HOMOTOPY EQUIVALENCE FOR NONPOSITIVELY CURVED MANIFOLDS

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ABSTRACT. Let M^n be a smooth closed manifold which admits a metric of nonpositive curvature. We show, using a theorem of Farrell and Hsiang, that if $n + k \geqslant 6$, then the surgery obstruction map $[M \times D^k, \partial; G/\mathrm{TOP}] \to L^h_{n+k}(\pi_1 M, w_1(M))$ is injective, where L^h_* are the obstruction groups for surgery up to homotopy equivalence.

Farrell and Hsiang have shown that if M^n is a closed nonpositively curved Riemannian manifold and $n + k \geqslant 6$, then the surgery map

$$[M^n \times D^k, \partial; G/\mathrm{TOP}, *] \xrightarrow{\theta_s} L^s_{n+k}(\pi_1 M, w_1(M))$$

for topological surgery in a split monomorphism **[1]**. This note considers the implications of their result of h-surgery, i.e. surgery up to homotopy equivalence rather than simple homotopy equivalence.

LEMMA. *Let Q^p be a compact manifold such that θ_s: $[Q^p, \partial; G/\mathrm{TOP}, *] \to L^s_p(\pi_1 Q, w_1(Q))$ is monic, let N^n be a compact, connected submanifold of Q^p which has a normal microbundle $\nu_Q(N)$, and let f: $U^p \to Q^p$ be a simple homotopy equivalence which is a homeomorphism near the boundary. Then any normal map $f|_V$: $V = f^{-1}(N) \to N$ induced by making f transverse to N is normally cobordant to a homeomorphism.*

PROOF. Recall that the long exact sequence of topological surgery

$$\to \mathcal{S}^a(Q^p \times D^k, \partial) \xrightarrow{\eta_a} [Q^p \times D^k, \partial; G/\mathrm{TOP}, *] \xrightarrow{\theta_a} L^a_{p+k}(\pi_1 Q, w_1(Q))$$

$$\xrightarrow{\partial} \mathcal{S}^a(Q^p \times D^{k-1}, \partial) \xrightarrow{\eta_a} \cdots \to [Q^p, \partial; G/\mathrm{TOP}, *] \xrightarrow{\theta_a} L^a_p(\pi_1 Q, w_1(Q))$$

(where $a = s$ or h denotes surgery up to simple homotopy equivalence or up to homotopy equivalence, respectively) is a long exact sequence of groups and group homomorphisms (see **[4]**; also see **[2]** for surgery in the topological category). Let $x = [f: U \to Q] \in \mathcal{S}^s(Q, \partial)$ and let i: $N \hookrightarrow Q$ be the inclusion map. Then the element $[f|_V: V \to N] \in [N, \partial; G/\mathrm{TOP}, *]$ defined above is $i^*(\eta_s(x))$, where i^*: $[Q, \partial; G/\mathrm{TOP}, *] \to [N, \partial; G/\mathrm{TOP}, *]$. Since θ_s is a monomorphism, image$(\eta_s) = 0$ in $[Q, \partial; G/\mathrm{TOP}, *]$, and since i^* is a homomorphism of groups we see that

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$i^*(\eta_s(x)) = i^*(0) = 0$, or that $i^*(\eta_s(x))$ is normally cobordant to a homeomorphism.

Geometrically, we have made the following construction. image$(\eta_s) = 0$ implies that each element x of the s-structure set $\mathcal{S}^s(Q, \partial)$ is obtained from a homeomorphism of Q by the action of an element of $L^s_{p+1}(\pi_1 Q, w_1(Q))$. That is, there is a normal map $[(X^{p+1}, \partial_0 X, \partial_1 X) \xrightarrow{g} (Q \times I, Q \times 0, Q \times 1)]$ so that $g|_{\partial_0 X}\colon \partial_0 X \to Q \times 0$ is a homeomorphism and $g|_{\partial_1 X}\colon \partial_1 X \to Q \times 1$ is a simple homotopy equivalence representing $x \in \mathcal{S}^s(Q, \partial)$. Make g transverse to $N \times I \subset Q \times I$ and let $Y = g^{-1}(N \times I)$. Then $[g^{-1}(N \times 1) \to N \times 1]$ represents $i^*(\eta_s(x))$ and $g|_Y\colon Y \to N \times I$ is the desired normal bordism.

PROPOSITION. *If M^n is a closed, nonpositively curved manifold then $\theta_h\colon [M \times D^k, \partial; G/\mathrm{TOP}, *] \to L^h_{n+k}(\pi_1 M, w_1(M))$ is split monic if $n + k \geqslant 6$.*

PROOF. We may use the Lemma to show that θ_h is monic, i.e. that image$(\eta_h) = 0$. Let $x \in \mathcal{S}^h(M \times D^k, \partial)$ be represented by a homotopy equivalence $f\colon (V^{n+k}, \partial) \to (M \times D^k, \partial)$ which is a homeomorphism near the boundary. Since taking products with the circle kills Whitehead torsion [**3**], $f \times \mathrm{Id}_{S^1}\colon V \times S^1 \to M \times D^k \times S^1$ is a simple homotopy equivalence. Since M is nonpositively curved, so is $M \times S^1$, and [**1**] shows that θ_s is monic for $M \times S^1$. Apply the Lemma with $Q = M \times S^1$ and $N = M \times 1$ to conclude that f is normally cobordant to a homeomorphism, i.e. $\eta_h(x) = 0$.

The splitting in [**1**] of θ_s induces a splitting of θ_h if we consider this commutative diagram:

$$\begin{array}{ccc} [M^n, \partial; G/\mathrm{TOP}, *] & \xrightarrow{\theta_h} & L^h_n(\pi_1 M, w_1(M)) \\ p^* \downarrow & & \downarrow \mathrm{tr}(p) \\ [M^n \times S^1, \partial; G/\mathrm{TOP}, *] & \xrightarrow{\theta_s} & L^s_{n+1}(\pi_1 M \times \mathbf{Z}, w_1(M) \circ \mathrm{pr}_1) \end{array}$$

Here $p\colon M^n \times S^1 \to M^n$ is a first-coordinate projection and $\mathrm{tr}(p)$ is the transfer associated to p [**4**, §4.2]. (The effect of $\mathrm{tr}(p)$ on one of the "objects" used in defining $L^h_n(\pi_1 M, w_1(M))$ in [**5**, Chapters 9 and 17D] is to multiply a map $(B, A) \to (Y, X)$ inducing a homotopy equivalence $A \to X$ by Id_{S^1}; by [**3**] this gives a map $(B \times S^1, A \times S^1) \to (Y \times S^1, X \times S^1)$ inducing a simple homotopy equivalence $A \times S^1 \to X \times S^1$, i.e. an "object" for the definition of $L^s_{n+1}(\pi_1 M \times \mathbf{Z}, w_1(M) \circ \mathrm{pr}_1)$.) If $i\colon M \to M \times S^1$ is the inclusion map, then i^* splits p^*. Compose i^* with the splitting of [**1**] for θ_s and with $\mathrm{tr}(p)$ to split θ_h.

More generally, the Proposition holds for any closed manifold M^n such that M^n and $M^n \times S^1$ both satisfy condition $(*)$ of [**1**, p. 201].

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