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ON THE DIFFERENTIABILITY OF FUNCTIONS IN $\mathbf{R}^n$

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SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

Recently, E. Stein [**2**] has extended the classical univariate differentiation theorem to functions defined on an open set $\Omega \subset \mathbf{R}^n$:

THEOREM (STEIN). *If the distribution gradient, ∇f, is locally in the Lorentz space L_{n1} on Ω, then f can be redefined on a set of measure zero so that f is continuous on Ω and*

$$f(x+h) - f(x) - \nabla f(x) \cdot h = o(h), \qquad h \to 0; \ a.e.\ x \in \Omega. \tag{1}$$

It can be shown [**3**, Chapter V, §6.3] that the space L_{n1} cannot be replaced by any larger Lorentz space and in this sense the theorem is sharp.

In this note, we shall prove Stein's Theorem using elementary principles. This should be compared with the proofs [**2** and **1**] which use more sophisticated Fourier analytic methods.

Since this is a local theorem, we may assume that $\Omega = Q_0$ is a cube in $\mathbf{R}^n$. Recall that a function ψ is in L_{n1} if and only if its decreasing rearrangement ψ^* (see [**4**, p. 189]) satisfies

$$\|\psi\|_{n1} := \int_0^\infty \psi^*(s) s^{1/n-1}\, ds < \infty. \tag{2}$$

LEMMA 1. *There is a constant $c > 0$ such that whenever the distributional gradient ∇f is in L_{n1}, Q is any cube in the interior of Q_0 and $f_Q := |Q|^{-1} \int_Q |f(y)|\, dy$, then*

$$|f_Q - f(x)| \leq c \int_Q |\nabla f(y)|\, |x-y|^{1-n}\, dy \leq c \|\chi_Q \nabla f\|_{n1}, \quad a.e.\ x \in Q. \tag{3}$$

PROOF. The right-hand inequality in (3) follows from the definition of the norm (2) and the Hardy-Littlewood inequality $|\int gh| \leq \int g^* h^*$ by taking $g = \nabla f$ and $h = |x - \cdot|^{1-n}$.

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For the left inequality in (3), we assume first that f is smooth. Any point $y \in Q$ can be written $y = x + \rho\sigma$ with $\sigma \in \Sigma$, the unit sphere in $\mathbf{R}^n$, and $\rho \le R(\sigma) \le \sqrt{n}|Q|^{1/n}$. We start with the one-variable inequality

$$|f(x+\rho\sigma) - f(x)| \le \left| \int_0^\rho \nabla f(x + r\sigma) \cdot \sigma \, dr \right| \le \int_0^\rho |\nabla f(x+r\sigma)| \, dr.$$

Averaging this inequality over Q in spherical coordinates and applying Fubini's theorem gives

$$\begin{aligned} |f_Q - f(x)| &\le \frac{1}{|Q|} \int_Q |f - f(x)| \\ &\le \frac{1}{|Q|} \int_\Sigma \int_0^{R(\sigma)} \int_0^\rho |\nabla f(x + r\sigma)| \, dr \rho^{n-1} \, d\rho \, d\sigma \\ &\le \int_\Sigma \int_0^{R(\sigma)} \left(\frac{1}{|Q|} \int_r^{R(\sigma)} \rho^{n-1} \, d\rho \right) (|\nabla f(x+r\sigma)| r^{1-n}) r^{n-1} \, dr \, d\sigma \\ &\le c \int_Q |\nabla f(y)| \, |x-y|^{1-n} \, dy. \end{aligned} \tag{4}$$

When f is not smooth, we approximate f by $f_\varepsilon := f * \phi_\varepsilon$ with ϕ in C_0^∞, $\phi \ge 0$ and $\int \phi = 1$. Then f_ε is defined for $x \in Q$ provided ε is sufficiently small and f_ε converges to f both in L_1 and a.e. Also $\nabla f_\varepsilon - \nabla f * \phi_\varepsilon$; so using the fact that (4) holds for f_ε and taking limits as $\varepsilon \to 0$ readily gives (3). □

LEMMA 2. *The function f can be redefined on a set of measure zero to be continuous in Q_0 and for any $x, x+h$ in the interior of Q_0,*

$$|f(x+h) - f(x)| \le c\|\chi_{Q(h)} \nabla f\|_{n1} \le c \int_0^{|h|^n} (\nabla f)^*(s) \; s^{1/n-1} \, ds \tag{5}$$

with $Q(h)$ any cube of side length $\le |h|$ which contains x and $x+h$.

PROOF. If $Q^* \subset Q$, then averaging (3) over Q^* gives

$$|f_Q - f_{Q^*}| \le c\|\chi_Q \nabla f\|_{n1} \le c \int_0^{|Q|} (\nabla f)^*(s) s^{1/n-1} \, ds. \tag{6}$$

The inequality (6) shows that $\{f_Q : Q \ni y\}$ is a Cauchy net and we redefine $f(y)$ to be its limit. By Lebesgue's theorem, this changes f at most on a set of measure zero. Taking a limit as $Q^* \downarrow \{y\}$ in (6) gives

$$|f_Q - f(y)| \le c\|\chi_Q(\nabla f)\|_{n1}, \quad y \in Q. \tag{7}$$

Now if $x,\ x+h \in Q(h)$, then using (7) with $Q = Q(h)$ for $y = x,\ x+h$ together with the triangle inequality gives (5). □

To establish the convergence (1), we use the maximal operator

$$\Lambda f(x) := \overline{\lim_{h \to 0}} \frac{|f(x+h) - f(x) - \nabla f(x) \cdot h|}{|h|}.$$

We show that $\Lambda f = 0$ a.e. by comparing Λf with $T(\nabla f)$ where

$$Tg(x) := \sup_{Q_0 \supset Q \ni x} \|\chi_Q g\|_{n1} / \|\chi_Q\|_{n1}. \tag{8}$$

The operator T was introduced in [**2**]. Since $\|\chi_E\|_{n1} = n|E|^{1/n}$ for measurable E, it follows that $T\chi_E = (M\chi_E)^{1/n}$ with M the Hardy-Littlewood maximal operator. Since M is of a weak type (1,1), T is of restricted weak type (n, n) and hence of weak type (n, n) (see Chapter V, §3 in [**4**]);

$$|\{Tg > \lambda\}| \le c(\lambda^{-1}\|g\|_{n1})^n, \qquad \lambda > 0. \tag{9}$$

LEMMA 3. *There is a constant c_0 depending only on n such that for each g with $\nabla g \in L_{n1}$, we have $\Lambda g \le c_0 T(\nabla g)$ a.e.*

PROOF. From (5) and the fact that $\|\chi_{Q(h)}\|_{n1} \le n|h|$, we have $|g(x+h) - g(x)| \le cT\nabla g(x)$. Since $|\nabla g| \le T(\nabla g)$ a.e., the lemma follows. □

PROOF OF THEOREM. Let $\lambda > 0$ and $E_\lambda := \{\Lambda f > \lambda\}$. If $\varepsilon > 0$, we let f_ε be a C^∞ function with $\|\nabla f - \nabla f_\varepsilon\|_{n1} < \varepsilon$. Then using (9) and Lemma 3 with $g_\varepsilon := f - f_\varepsilon$, we have

$$\begin{aligned}|E_\lambda| = |\{\Lambda g_\varepsilon > \lambda\}| &\le |\{T(\nabla g_\varepsilon) > \lambda/c_0\}| \\ &\le c(c_0\|\nabla g_\varepsilon\|_{n1}/\lambda)^n \le c(c_0\varepsilon/\lambda)^n.\end{aligned}$$

Letting $\varepsilon \to 0$ shows that $|E_\lambda| = 0$. Hence $\Lambda f = 0$ a.e. □

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