APPLICATIONS OF THE
JOINT ANGULAR FIELD OF VALUES
GEORGE PHILLIP BARKER

ABSTRACT. Let $A_1, \ldots, A_m$ be $n \times n$ hermitian matrices and let $\mathcal{H}_n$ be the real space of $n \times n$ hermitian matrices. If $\text{span}\{A_1, \ldots, A_m\} = \mathcal{H}_n$, then the extreme rays of the joint angular field of values of $\{A_1, \ldots, A_m\}$ are determined. Then this cone is used to give necessary and sufficient conditions for the existence of hermitian matrices $B_1, \ldots, B_m$ such that $A_1 \otimes B_1 + \cdots + A_m \otimes B_m$ preserves the cone of the positive semidefinite matrices where $A \otimes B$ is the dyad product $A \otimes B(H) = (\text{tr} BH)A$.

1. Introduction. Let $V$ be a finite-dimensional real vector space of dimension $n$ with an inner product $(\cdot, \cdot)$. A subset $K \subset V$ is called a (convex) cone iff for all $\alpha, \beta \geq 0$ and $x, y \in K$ we have $\alpha x + \beta y \in K$. We call $K$ closed if it is a closed set in the natural topology of $V$. The cone $K$ is pointed iff $K \cap (-K) = \{0\}$ (i.e., $K$ contains no nonzero subspace) and full iff $K$ has nonempty interior. In finite-dimensional spaces it is well known that $K$ is full iff $K - K = V$. A face $F$ of a cone $K$ is a subcone such that $x, y - x \in K$ and $y \in F$ implies $x \in F$. The dimension of a face $F$, $\dim F$, is the dimension of the linear span of $F$. If $\dim F = 1$ we call $F$ an extreme ray of $K$. If $S \subset K$, then $\Phi(S)$ denotes the least face (with respect to inclusion) containing $S$. When $S = \{x\}$, we write simply $\Phi(x)$, and if $\Phi(x)$ is an extreme ray we call $x$ an extremal.

With any cone $K$ we associate two special cones (cf. [8, Chapter 7]). Set

$$K' = \{y \in V : (y, x) \geq 0, \forall x \in K\}.$$

Next if $\text{Hom}(V)$ denotes the vector space of linear mappings of $V$, then define

$$\Pi(K) = \{A \in \text{Hom}(V) : AK \subset K\}.$$

Let $M_n(C)$ denote the vector space of $n \times n$ matrices with complex entries and $C^n$ the space of column vectors. Denote by $A^*$ or $x^*$ the conjugate transpose of $A$ or $x$. $M_n(C)$ is an inner product space under $(A, B) = \text{tr} B^* A$. If $A$ and $B$ are hermitian matrices, then $(A, B)$ is real. Thus the set $\mathcal{H}_n$ of hermitian matrices is a real inner product space. The set $\mathcal{P}_n$ of positive semidefinite matrices is a closed pointed full cone in $\mathcal{H}_n$ [1]. In fact, $\mathcal{P}_n$ is self-dual; that is, $\mathcal{P}_n' = \mathcal{P}_n$, and the faces of $\mathcal{P}_n$ are known [1, 3]. In particular, the extreme rays are the faces determined by rank one orthogonal projections. The identity matrix $I$ is in the interior of $\mathcal{P}_n'$ so that

$$\mathcal{B} = \{P \in \mathcal{P}_n : \text{tr} P = 1 = (I, P)\}$$
is a compact convex cross section of $\mathcal{P}_n$. That is, $\mathcal{B}$ is a compact convex subset of $\mathcal{P}_n$ and for every nonzero $P \in \mathcal{P}_n$ there is a unique $\alpha > 0$ such that $\alpha P \in \mathcal{B}$.

An as yet unsolved problem is to determine the set of all linear transformations of $\mathbb{C}^n$ which leave $\mathcal{P}_n$ invariant, that is, describe $\Pi(\mathcal{P}_n)$ (cf. [5 and 6]). Work on this problem led to an extension of the classical field of values for a single complex matrix to a kind of joint field of values for a set of complex matrices (cf. [2] and §3). In the application we shall take the matrices to be hermitian, but this is not necessary (cf. §2).

Recall that the classical (Hausdorff-Topelitz) field of values for $A \in M_n(\mathbb{C})$ is the set

$$W(A) = \{x^*Ax : x^*x = 1, x \in \mathbb{C}^n\}.$$  

$W(A)$ is a compact convex subset of $\mathbb{C}$, and if $A$ is hermitian then $W(A)$ is an interval on the real axis.

**Proposition 1.** $W(A) = \{\text{tr} PA : P \in \mathcal{B}\}$.

**Proof.** Since the trace is a continuous linear functional and $\mathcal{B}$ is a compact set, then $\{\text{tr} PA : P \in \mathcal{B}\}$ is a compact convex set which contains $W(A)$.

On the other hand each $P \in \mathcal{B}$ is a convex combination of the extreme points of $\mathcal{B}$. Since $\mathcal{B}$ is a cross section of $\mathcal{P}_n$ determined by a positive linear functional, the extreme points of $\mathcal{B}$ are the intersection of the extreme rays of $\mathcal{P}_n$ with the affine hyperplane

$$\{H \in \mathcal{H}_n : (H, I) = 1\}.$$  

If $P$ is such an extreme point, then $P$ is a nonnegative multiple of a rank one orthogonal projection, say $P = \alpha xx^*$ where $x^*x = 1$. But since $P \in \mathcal{B}$, then $1 = (P, I) = \alpha x^*x = \alpha$. Thus for any $Q \in \mathcal{B}$ we have

$$Q = \sum_j \alpha_j x_j x_j^*$$

where for all $j$, $x_j^*x_j = 1$, $\alpha_j \geq 0$, and $\sum \alpha_j = 1$. Thus for any $a \in M_n(\mathbb{C})$,

$$\text{tr} QA = \text{tr} \left( \sum_j \alpha_j x_j x_j^* A \right) = \sum_j \alpha_j x_j^* A x_j \in W(A).$$

The equality follows.

2. **The joint angular field of values.** Let $A_1, \ldots, A_m$ be $n \times n$ complex matrices. The set

$$\{(x^*A_1x, \ldots, x^*A_mx) : x^*x = 1\}$$

is usually taken to be the joint spacial numerical range of the $A_j$ [4, p. 137]. It is known [4, p. 138] that for $m \geq 2$ this set is generally not convex. In [4] it is asserted that for hermitian matrices the joint spacial numerical range is a convex subset of $\mathbb{R}^n$. Unfortunately, this remark is false as the example on p. 138 of [4] can be used to show. For if $A, B$ are complex matrices with Toeplitz decompositions $A = H + iK$, $B = L + iM$ in which $H, K, L, M$ are hermitian, then the convexity of the joint numerical range of $H, K, L, M$ in $\mathbb{R}^4$ entails the convexity of the joint
numerical range of $A$ and $B$ in $C^2$. Direct numerical examples can also be given. Motivated by Proposition 1 we define the joint field of values of $A_1, \ldots, A_m$ to be

$$W(A_1, \ldots, A_m) = \{(\text{tr} PA_1, \ldots, \text{tr} PA_m) : P \in \mathcal{B}\}.$$ 

This set is a compact convex set and is the convex hull of the joint numerical range mentioned previously. It is also related to the $k$-numerical ranges of Fillmore and Williams [7].

In the remainder of the paper we shall be concerned with applications to hermitian matrices, so from this point we assume $A_1, \ldots, A_m \in \mathcal{H}_n$. Thus $W(A_1, \ldots, A_m) \subset \mathbb{R}^m$. Let $J(A_1, \ldots, A_m)$ be the cone in $\mathbb{R}^m$ generated by $W(A_1, \ldots, A_m)$. When $0 \notin W(A_1, \ldots, A_m)$ then (e.g. [9, Lemma 1]) the cone $J(A_1, \ldots, A_m) = \{av : a \geq 0, v \in W(A_1, \ldots, A_m)\}$ is closed. We call $J(A_1, \ldots, A_m)$ the joint angular field of values of $A_1, \ldots, A_m$. In [2] this set was defined using a different cross section $\mathcal{B}_1$ of $\mathcal{P}_n$. However, since $\mathcal{B}$ and $\mathcal{B}_1$ are both cross sections of $\mathcal{P}_n$, it is easy to check that they give rise to the same angular field of values.

**Notation.** Let $A_1, \ldots, A_m \in \mathcal{H}_n$. When the matrices are clear from the context, we shall write $J$ for $J(A_1, \ldots, A_m)$, $W$ for $W(A_1, \ldots, A_m)$, and let $S = \text{span}\{A_1, \ldots, A_m\}$.

From [2] we have the following result.

**Theorem 2.** Let $A_1, \ldots, A_m \in \mathcal{H}_n$ with $0 \notin W$. The extreme rays of $J$ are among the rays determined by the points $(x^* A_1 x, \ldots, x^* A_m x)$ where $x^* x = 1$, $x \in C^n$. Furthermore, if $\{A_1, \ldots, A_m\}$ spans $\mathcal{H}_n$, then $J$ is pointed; $J$ is full iff $\{A_1, \ldots, A_m\}$ is linearly independent.

In order to discuss the extreme rays of $J$ we want them "all to be there", that is, we want $J$ to be closed. The hypothesis $0 \notin W$ in Theorem 2 will insure this. This condition will also occur in Theorem 3. However, the closure of $J$ is not needed for Theorem 6 of [2] (which is restated below as Theorem 4), and so it is not assumed in Theorem 5 either. We can refine somewhat our knowledge of the extremal structure of the angular field of values. For convenience we take $m > 1$ and $n > 1$.

**Theorem 3.** Let $m > 1$, $n > 1$ and $A_1, \ldots, A_m \in \mathcal{H}_n$. Then $0 \notin W$ iff $S \cap \text{int} \mathcal{P}_n \neq \emptyset$. Furthermore, if $S = \mathcal{H}_n$, then every vector of the form $(\text{tr} PA_1, \ldots, \text{tr} PA_m)$, where $P \in \mathcal{P}_n$, is of rank one, is an extremal of $J$. 

**Proof.** Let $L$ be a subspace of $\mathbb{R}^m$ and let $K$ be a closed pointed full cone in $\mathbb{R}^m$. An easy extension of the Gordan-Stiemke theorem (cf. [8]) asserts that $L \cap K \neq \emptyset$ iff $L^\perp \cap K^\prime = \{0\}$. If we take $K = \mathcal{P}_n$ which is self-dual and $L = S$, then we have

$$S \cap \text{int} \mathcal{P}_n \neq \emptyset \quad \text{iff} \quad S^\perp \cap \mathcal{P}_n = \{0\}.$$ 

If $0 \notin W$, then clearly $S^\perp \cap \mathcal{P}_n = \{0\}$. Conversely, if $S \cap \text{int} \mathcal{P}_n \neq \emptyset$, then for some real scalars $\alpha_1, \ldots, \alpha_m$ we have $\sum \alpha_j A_j \in \text{int} \mathcal{P}_n$. But if $0 \in W$, then for some $P \in \mathcal{P}_n$, $P \neq 0$, we have $(\text{tr} PA_1, \ldots, \text{tr} PA_m) = 0$. But then

$$\sum \alpha_j \text{tr} PA_j = \left\langle P, \sum \alpha_j A_j \right\rangle = 0,$$

which cannot happen since $\sum \alpha_j A_j$ is positive definite.
Now suppose \( S = \mathcal{K}_n \). From Theorem 2 we know that every extremal is of the desired form, so it suffices to show that every vector of the form \( z = (\text{tr} PA_1, \ldots, \text{tr} PA_m) \) with rank \( P = 1 \) is an extremal. Suppose for some \( z \) as above there exist \( P_1, P_2 \in \mathcal{P}_n \) with \( z_j = (\text{tr} P_j A_1, \ldots, \text{tr} P_j A_m) \) and \( z = z_1 + z_2 \). But then \( P - (P_1 + P_2) \in S^\perp = \{0\} \), so that \( P = P_1 + P_2 \). Since \( P \) is an extremal of \( \mathcal{P}_n \), we have \( P_j = \alpha_j P \), \( j = 1,2 \), where \( \alpha_j \geq 0 \). Thus \( z \) must be an extremal of \( J \).

Example. Let

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{ and } \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then \( J(A_1, A_2) = \{(x_1, x_2) : x_j \geq 0\} \). Each extremal of \( J \) is of the appropriate form, but \( \text{span}\{A_1, A_2\} \neq \mathcal{K}_2 \).

3. Applications to \( \Pi(\mathcal{P}_n) \). Recall from the introduction that \( \Pi(\mathcal{P}_n) \) is the set of all (real) linear transformation \( T \) of \( \mathcal{K}_n \) such that \( T(\mathcal{P}_n) \subset \mathcal{P}_n \). Following [2 and 8] we can represent each \( T \in \text{Hom}(\mathcal{K}_n) \) by a tensor product. Specifically, we use \( H \otimes K \) to denote the dyad rather than the Kronecker product, so that for \( H, K, P \in \mathcal{K}_n \) we have \( H \otimes K(P) = (\text{tr} KP)H \). Therefore for each \( T \in \text{Hom}(\mathcal{K}_n) \), there are \( H_1, \ldots, H_m, K_1, \ldots, K_m \in \mathcal{K}_n \) such that \( T = \sum_j H_j \otimes K_j \), and \( T \in \Pi(\mathcal{P}_n) \) iff for all \( P \in \mathcal{P}_n \)

\[
T(P) = \sum_j (\text{tr} K_j P)H_j \in \mathcal{P}_n.
\]

We state from [2] the connection with the joint angular field of values.

**Theorem 4.** Let \( H_1, \ldots, H_m, K_1, \ldots, K_m \in \mathcal{K}_n \). Then

\[
T = \sum H_j \otimes K_j \in \Pi(\mathcal{P}_n) \iff J(K_1, \ldots, K_m) \subset J(H_1, \ldots, H_m)'.
\]

If \( T = \sum H_j \otimes K_j \), we may renumber the matrices so that \( \{H_1, \ldots, H_t\} \) is a maximal linearly independent subset of \( \{H_1, \ldots, H_m\} \). Then we can represent \( T \) as

\[
T = \sum_{j=1}^t H_j \otimes L_j
\]

where the \( L_j \) are obvious real linear combinations of the \( K_i \) which are also hermitian matrices. Thus it is reasonable to restrict ourselves to linearly independent sets of hermitian matrices. Recall that \( S = \text{span}\{A_1, \ldots, A_m\} \).

**Theorem 5.** Let \( m > 1, n > 1 \) and let \( A_1, \ldots, A_m \) be linearly independent elements of \( \mathcal{K}_n \). Then there are \( B_1, \ldots, B_m \in \mathcal{K}_n \) not all zero such that

\[
T = \sum A_j \otimes B_j \in \Pi(\mathcal{P}_n) \iff S \cap \mathcal{P}_n \neq \{0\}.
\]

**Lemma.** Let \( A_1, \ldots, A_m \in \mathcal{K}_n \). If \( J(A_1, \ldots, A_m)' = \{0\} \), then \( S \cap \mathcal{P}_n = \{0\} \). If \( A_1, \ldots, A_m \) are linearly independent and \( S \cap \mathcal{P}_n = \{0\} \), then \( J(A_1, \ldots, A_m)' = \{0\} \).

**Discussion.** For \( A_1, \ldots, A_m \in \mathcal{K}_n \) with the corresponding \( S, W, J \) and \( J' \), note that \( J' = \{0\} \) iff the closure of \( J \) is \( \mathbb{R}^m \). Since \( J \) is a convex set in a finite-dimensional space, this is equivalent to \( J = \mathbb{R}^m \), which is equivalent to \( 0 \in \text{int} W \).
Thus the hypotheses of the Lemma are not exactly the same as those of Theorem 3 where we are concerned with $S \cap \text{int} \mathcal{P}_n$.

**Proof.**

\[(y_1, \ldots, y_m) \in J(A_1, \ldots, A_m)' \iff y_1 \text{tr} P A_1 + \cdots + y_m \text{tr} P A_m \geq 0 \quad \forall P \in \mathcal{P}_n
\]
\[\iff \text{tr} [P(y_1 A_1 + \cdots + y_m A_m)] \geq 0 \quad \forall P \in \mathcal{P}_n
\]
\[\iff y_1 A_1 + \cdots + y_m A_m \in S \cap \mathcal{P}_n.
\]

Thus if $S \cap \mathcal{P}_n \not= \{0\}$, then $J(A_1, \ldots, A_m)' \not= \{0\}$. On the other hand, if $S \cap \mathcal{P}_n = \{0\}$ and $A_1, \ldots, A_m$ are linearly independent, then $y_1 A_1 + \cdots + y_m A_m \in \mathcal{P}_n$ implies $y_1 = \cdots = y_m = 0$, so that $J(A_1, \ldots, A_m)' = \{0\}$.

**Proof of Theorem 5.** If there are $B_1, \ldots, B_m$ not all zero such that $\sum A_j \otimes B_j \in \Pi(\mathcal{P}_n)$, then $J(B_1, \ldots, B_m) \not= \{0\}$, so by Theorem 4 $J(A_1, \ldots, A_m)' \not= \{0\}$. Thus from the Lemma $S \cap \mathcal{P}_n \not= \{0\}$.

Conversely, suppose $\sum y_j A_j \in S \cap \mathcal{P}_n$, $\sum y_j A_j \not= 0$. Then not all the $y_j$ are zero. Let $B_j = y_j I$. For any $P \in \mathcal{B}$ we have $\text{tr} P = 1$, so that

\[\sum A_j \otimes B_j(P) = \sum y_j A_j \in \mathcal{P}_n.
\]

Since every nonzero element of $\mathcal{P}_n$ is a positive multiple of some $P \in \mathcal{B}$, the result follows.

**References**


Department of Mathematics, University of Missouri, Kansas City, Missouri 64110