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APPLICATIONS OF THE JOINT ANGULAR FIELD OF VALUES

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ABSTRACT. Let $A_1, \ldots, A_m$ be $n \times n$ hermitian matrices and let $\mathcal{H}_n$ be the real space of $n \times n$ hermitian matrices. If $\operatorname{span}\{A_1, \ldots, A_m\} = \mathcal{H}_n$, then the extreme rays of the joint angular field of values of $\{A_1, \ldots, A_m\}$ are determined. Then this cone is used to give necessary and sufficient conditions for the existence of hermitian matrices $B_1, \ldots, B_m$ such that $A_1 \otimes B_1 + \cdots + A_m \otimes B_m$ preserves the cone of the positive semidefinite matrices where $A \otimes B$ is the dyad product $A \otimes B(H) = (\operatorname{tr} BH)A$.

1. Introduction. Let V be a finite-dimensional real vector space of dimension n with an inner product $(\ ,\ )$. A subset $K \subset V$ is called a (convex) *cone* iff for all $\alpha, \beta \geq 0$ and $x, y \in K$ we have $\alpha x + \beta y \in K$. We call K *closed* if it is a closed set in the natural topology of V. The cone K is *pointed* iff $K \cap (-K) = \{0\}$ (i.e., K contains no nonzero subspace) and *full* iff K has nonempty interior. In finite-dimensional spaces it is well known that K is full iff $K - K = V$. A *face* F of a cone K is a subcone such that $x, y - x \in K$ and $y \in F$ implies $x \in F$. The dimension of a face $F, \dim F$, is the dimension of the linear span of F. If $\dim F = 1$ we call F an extreme ray of K. If $S \subset K$, then $\phi(S)$ denotes the least face (with respect to inclusion) containing S. When $S = \{x\}$, we write simply $\phi(x)$, and if $\phi(x)$ is an extreme ray we call x an *extremal.*

With any cone K we associate two special cones (cf. [**8**, Chapter 7]). Set

$$K' = \{y \in V : (y, x) \geq 0,\ \forall x \in K\}.$$

Next if $\operatorname{Hom}(V)$ denotes the vector space of linear mappings of V, then define

$$\Pi(K) = \{A \in \operatorname{Hom}(V) : AK \subset K\}.$$

Let $M_n(\mathbf{C})$ denote the vector space of $n \times n$ matrices with complex entries and $\mathbf{C}^n$ the space of column vectors. Denote by A^* or x^* the conjugate transpose of A or x. $M_n(\mathbf{C})$ is an inner product space under $(A, B) = \operatorname{tr} B^* A$. If A and B are hermitian matrices, then (A, B) is real. Thus the set $\mathcal{H}_n$ of hermitian matrices is a real inner product space. The set $\mathcal{P}_n$ of positive semidefinite matrices is a closed pointed full cone in $\mathcal{H}_n$ [**1**]. In fact, $\mathcal{P}_n$ is self-dual; that is, $\mathcal{P}_n' = \mathcal{P}_n$, and the faces of $\mathcal{P}_n$ are known [**1, 3**]. In particular, the extreme rays are the faces determined by rank one orthogonal projections. The identity matrix I is in the interior of $\mathcal{P}_n' = \mathcal{P}_n$ so that

$$\mathcal{B} = \{P \in \mathcal{P}_n : \operatorname{tr} P = 1 = (I, P)\}$$

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is a compact convex cross section of $\mathcal{P}_n$. That is, $\mathcal{B}$ is a compact convex subset of $\mathcal{P}_n$ and for every nonzero $P \in \mathcal{P}_n$ there is a unique $\alpha > 0$ such that $\alpha P \in \mathcal{B}$.

An as yet unsolved problem is to determine the set of all linear transformations of $\mathcal{H}_n$ which leave $\mathcal{P}_n$ invariant, that is, describe $\Pi(\mathcal{P}_n)$ (cf. [**5** and **6**]). Work on this problem led to an extension of the classical field of values for a single complex matrix to a kind of joint field of values for a set of complex matrices (cf. [**2**] and §3). In the application we shall take the matrices to be hermitian, but this is not necessary (cf. §2).

Recall that the classical (Hausdorff-Topelitz) field of values for $A \in M_n(\mathbf{C})$ is the set

$$W(A) = \{x^*Ax : x^*x = 1,\ x \in \mathbf{C}^n\}.$$

$W(A)$ is a compact convex subset of $\mathbf{C}$, and if A is hermitian then $W(A)$ is an interval on the real axis.

PROPOSITION 1. $W(A) = \{\operatorname{tr} PA : P \in \mathcal{B}\}$.

PROOF. Since the trace is a continuous linear functional and $\mathcal{B}$ is a compact set, then $\{\operatorname{tr} PA : P \in \mathcal{B}\}$ is a compact convex set which contains $W(A)$.

On the other hand each $P \in \mathcal{B}$ is a convex combination of the extreme points of $\mathcal{B}$. Since $\mathcal{B}$ is a cross section of $\mathcal{P}_n$ determined by a positive linear functional, the extreme points of $\mathcal{B}$ are the intersection of the extreme rays of $\mathcal{P}_n$ with the affine hyperplane

$$\{H \in \mathcal{H}_n : (H, I) = 1\}.$$

If P is such an extreme point, then P is a nonnegative multiple of a rank one orthogonal projection, say $P = \alpha x x^*$ where $x^*x = 1$. But since $P \in \mathcal{B}$, then $1 = (P, I) = \alpha x^* x = \alpha$. Thus for any $Q \in \mathcal{B}$ we have

$$Q = \sum_j \alpha_j x_j x_j^*$$

where for all j, $x_j^* x_j = 1$, $\alpha_j \geq 0$, and $\sum \alpha_j = 1$. Thus for any $a \in M_n(\mathbf{C})$,

$$\operatorname{tr} QA = \operatorname{tr}\left[\left(\sum_j \alpha_j x_j x_j^*\right) A\right] = \sum_j \alpha_j x_j^* A x_j \in W(A).$$

The equality follows.

2. The joint angular field of values. Let $A_1, \ldots, A_m$ be $n \times n$ complex matrices. The set

$$\{(x^*A_1x, \ldots, x^*A_mx) : x^*x = 1\}$$

is usually taken to be the joint spacial numerical range of the A_j [**4**, p. 137]. It is known [**4**, p. 138] that for $m \geq 2$ this set is generally not convex. In [**4**] it is asserted that for hermitian matrices the joint spacial numerical range is a convex subset of $\mathbf{R}^n$. Unfortunately, this remark is false as the example on p. 138 of [**4**] can be used to show. For if A, B are complex matrices with Toeplitz decompositions $A = H + iK$, $B = L + iM$ in which H, K, L, M are hermitian, then the convexity of the joint numerical range of H, K, L, M in $\mathbf{R}^4$ entails the convexity of the joint numerical range of A and B in $\mathbf{C}^2$. Direct numerical examples can also be given. Motivated by Proposition 1 we define the *joint field of values* of $A_1, \dots, A_m$ to be

$$W(A_1, \dots, A_m) = \{(\operatorname{tr} PA_1, \dots, \operatorname{tr} PA_m) \colon P \in \mathcal{B}\}.$$

This set is a compact convex set and is the convex hull of the joint numerical range mentioned previously. It is also related to the k-numerical ranges of Fillmore and Williams [**7**].

In the remainder of the paper we shall be concerned with applications to hermitian matrices, so from this point we assume $A_1, \dots, A_m \in \mathcal{H}_n$. Thus $W(A_1, \dots, A_m) \subset \mathcal{R}^m$. Let $J(A_1, \dots, A_m)$ be the cone in $\mathbf{R}^m$ generated by $W(A_1, \dots, A_m)$. When $0 \notin W(A_1, \dots, A_m)$ then (e.g. [**9**, Lemma 1]) the cone $J(A_1, \dots, A_m) = \{\alpha v \colon \alpha \geq 0,\ v \in W(A_1, \dots, A_m)\}$ is closed. We call $J(A_1, \dots, A_m)$ the *joint angular field of values* of $A_1, \dots, A_m$. In [**2**] this set was defined using a different cross section $\mathcal{B}_1$ of $\mathcal{P}_n$. However, since $\mathcal{B}$ and $\mathcal{B}_1$ are both cross sections of $\mathcal{P}_n$, it is easy to check that they give rise to the same angular field of values.

NOTATION. Let $A_1, \dots, A_m \in \mathcal{H}_n$. When the matrices are clear from the context, we shall write J for $J(A_1, \dots, A_m)$, W for $W(A_1, \dots, A_m)$, and let $S = \operatorname{span}\{A_1, \dots, A_m\}$.

From [**2**] we have the following result.

THEOREM 2. *Let $A_1, \dots, A_m \in \mathcal{H}_n$ with $0 \notin W$. The extreme rays of J are among the rays determined by the points $(x^*A_1x, \dots, x^*A_mx)$ where $x^*x = 1$, $x \in \mathbf{C}^n$. Furthermore, if $\{A_1, \dots, A_m\}$ spans $\mathcal{H}_n$, then J is pointed; J is full iff $\{A_1, \dots, A_m\}$ is linearly independent.*

In order to discuss the extreme rays of J we want them "all to be there", that is, we want J to be closed. The hypothesis $0 \notin W$ in Theorem 2 will insure this. This condition will also occur in Theorem 3. However, the closure of J is not needed for Theorem 6 of [**2**] (which is restated below as Theorem 4), and so it is not assumed in Theorem 5 either. We can refine somewhat our knowledge of the extremal structure of the angular field of values. For convenience we take $m > 1$ and $n > 1$.

THEOREM 3. *Let $m > 1$, $n > 1$ and $A_1, \dots, A_m \in \mathcal{H}_n$. Then $0 \notin W$ iff $S \cap \operatorname{int} \mathcal{P}_n \neq \emptyset$. Furthermore, if $S = \mathcal{H}_n$, then every vector of the form $(\operatorname{tr} PA_1, \dots, \operatorname{tr} PA_m)$, where $P \in \mathcal{P}_n$ is of rank one, is an extremal of J.*

PROOF. Let L be a subspace of $\mathbf{R}^m$ and let K be a closed pointed full cone in $\mathbf{R}^m$. An easy extension of the Gordan-Stiemke theorem (cf. [**8**]) asserts that $L \cap \operatorname{int} K \neq \emptyset$ iff $L^\perp \cap K' = \{0\}$. If we take $K = \mathcal{P}_n$ which is self-dual and $L = S$, then we have

$$S \cap \operatorname{int} \mathcal{P}_n \neq \emptyset \quad \text{iff } S^\perp \cap \mathcal{P}_n = \{0\}.$$

If $0 \notin W$, then clearly $S^\perp \cap \mathcal{P}_n = \{0\}$. Conversely, if $S \cap \operatorname{int} \mathcal{P}_n \notin \emptyset$, then for some real scalars $\alpha_1, \dots \alpha_m$ we have $\sum \alpha_j A_j \in \operatorname{int} \mathcal{P}_n$. But if $0 \in W$, then for some $P \in \mathcal{P}_n$, $P \neq 0$, we have $(\operatorname{tr} PA_1, \dots, \operatorname{tr} PA_m) = 0$. But then

$$\sum \alpha_j \operatorname{tr} PA_j = \left\langle P, \sum \alpha_j A_j \right\rangle = 0,$$

which cannot happen since $\sum \alpha_j A_j$ is positive definite.

Now suppose $S = \mathcal{H}_n$. From Theorem 2 we know that every extremal is of the desired form, so it suffices to show that every vector of the form $z = (\operatorname{tr} PA_1, \ldots, \operatorname{tr} PA_m)$ with rank $P = 1$ is an extremal. Suppose for some z as above there exist $P_1, P_2 \in \mathcal{P}_n$ with $z_j = (\operatorname{tr} P_j A_1, \ldots, \operatorname{tr} P_j A_m)$ and $z = z_1 + z_2$. But then $P - (P_1 + P_2) \in S^\perp = \{0\}$, so that $P = P_1 + P_2$. Since P is an extremal of $\mathcal{P}_n$, we have $P_j = \alpha_j P$, $j = 1, 2$, where $\alpha_j \geq 0$. Thus z must be an extremal of J.

EXAMPLE. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $J(A_1, A_2) = \{(x_1, x_2) : x_j \geq 0\}$. Each extremal of J is of the appropriate form, but $\operatorname{span}\{A_1, A_2\} \neq \mathcal{H}_2$.

3. Applications to $\Pi(\mathcal{P}_n)$. Recall from the introduction that $\Pi(\mathcal{P}_n)$ is the set of all (real) linear transformation T of $\mathcal{H}_n$ such that $T(\mathcal{P}_n) \subset \mathcal{P}_n$. Following [**2** and **8**] we can represent each $T \in \operatorname{Hom}(\mathcal{H}_n)$ by a tensor product. Specifically, we use $H \otimes K$ to denote the dyad rather than the Kronecker product, so that for $H, K, P \in \mathcal{H}_n$ we have $H \otimes K(P) = (\operatorname{tr} KP)H$. Therefore for each $T \in \operatorname{Hom}(\mathcal{H}_n)$, there are $H_1, \ldots, H_m$, $K_1, \ldots, K_m \in \mathcal{H}_n$ such that $T = \sum_j H_j \otimes K_j$, and $T \in \Pi(\mathcal{P}_n)$ iff for all $P \in \mathcal{P}_n$

$$T(P) = \sum_j (\operatorname{tr} K_j P) H_j \in \mathcal{P}_n.$$

We state from [**2**] the connection with the joint angular field of values.

THEOREM 4. *Let $H_1, \ldots, H_m$, $K_1, \ldots, K_m \in \mathcal{H}_n$. Then*

$$T = \sum H_j \otimes K_j \in \Pi(\mathcal{P}_n) \quad \textit{iff } J(K_1, \ldots, K_m) \subset J(H_1, \ldots, H_m)'.$$

If $T = \sum H_j \otimes K_j$, we may renumber the matrices so that $\{H_1, \ldots H_t\}$ is a maximal linearly independent subset of $\{H_1, \ldots, H_m\}$. Then we can represent T as

$$T = \sum_{j=1}^{t} H_j \otimes L_j$$

where the L_j are obvious real linear combinations of the K_i which are also hermitian matrices. Thus it is reasonable to restrict ourselves to linearly independent sets of hermitian matrices. Recall that $S = \operatorname{span}\{A_1, \ldots, A_m\}$.

THEOREM 5. *Let $m > 1$, $n > 1$ and let $A_1, \ldots, A_m$ be linearly independent elements of $\mathcal{H}_n$. Then there are $B_1, \ldots, B_m \in \mathcal{H}_n$ not all zero such that*

$$T = \sum A_j \otimes B_j \in \Pi(\mathcal{P}_n) \quad \textit{iff } S \cap \mathcal{P}_n \neq \{0\}.$$

LEMMA. *Let $A_1, \ldots, A_m \in \mathcal{H}_n$. If $J(A_1, \ldots, A_m)' = \{0\}$, then $S \cap \mathcal{P}_n = \{0\}$. If $A_1, \ldots, A_m$ are linearly independent and $S \cap \mathcal{P}_n = \{0\}$, then $J(A_1, \ldots, A_m)' = \{0\}$.*

DISCUSSION. For $A_1, \ldots, A_m \in \mathcal{H}_n$ with the corresponding S, W, J and J', note that $J' = \{0\}$ iff the closure of J is $\mathbf{R}^m$. Since J is a convex set in a finite-dimensional space, this is equivalent to $J = \mathbf{R}^m$, which is equivalent to $0 \in \operatorname{int} W$.

Thus the hypotheses of the Lemma are not exactly the same as those of Theorem 3 where we are concerned with $S \cap \operatorname{int} \mathcal{P}_n$.

PROOF.

$$\begin{aligned}(y_1, \ldots, y_m) \in J(A_1, \ldots, A_m)' &\Leftrightarrow y_1 \operatorname{tr} PA_1 + \cdots + y_m \operatorname{tr} PA_m \geq 0 \quad \forall P \in \mathcal{P}_n \\ &\Leftrightarrow \operatorname{tr}[P(y_1A_1 + \cdots + y_mA_m)] \geq 0 \quad \forall P \in \mathcal{P}_n \\ &\Leftrightarrow y_1A_1 + \cdots + y_mA_m \in S \cap \mathcal{P}_n.\end{aligned}$$

Thus if $S \cap \mathcal{P}_n \neq \{0\}$, then $J(A_1, \ldots, A_m)' \neq \{0\}$. On the other hand, if $S \cap \mathcal{P}_n = \{0\}$ and $A_1, \ldots, A_m$ are linearly independent, then $y_1A_1 + \cdots + y_mA_m \in \mathcal{P}_n$ implies $y_1 = \cdots = y_m = 0$, so that $J(A_1, \ldots, A_m)' = \{0\}$.

PROOF OF THEOREM 5. If there are $B_1, \ldots, B_m$ not all zero such that $\sum A_j \otimes B_j \in \Pi(\mathcal{P}_n)$, then $J(B_1, \ldots, B_m) \neq \{0\}$, so by Theorem 4 $J(A_1, \ldots, A_m)' \neq \{0\}$. Thus from the Lemma $S \cap \mathcal{P}_n \neq \{0\}$.

Conversely, suppose $\sum y_jA_j \in S \cap \mathcal{P}_n$, $\sum y_jA_j \neq 0$. Then not all the y_j are zero. Let $B_j = y_jI$. For any $P \in \mathcal{B}$ we have $\operatorname{tr} P = 1$, so that

$$\sum A_j \otimes B_j(P) = \sum y_jA_j \in \mathcal{P}_n.$$

Since every nonzero element of $\mathcal{P}_n$ is a positive multiple of some $P \in \mathcal{B}$, the result follows.

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