ON MAXIMA OF TAKAGI-VAN DER WAERDEN FUNCTIONS
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ABSTRACT. Generalizing Takagi’s function $F_2(x)$ and van der Waerden’s function $F_{10}(x)$, we introduce a class of nowhere differentiable continuous functions $F_r(x)$, $r \geq 2$. Some properties of $F_r(x)$ concerning especially maxima are discussed. When $r$ is even, the Hausdorff dimension of the set of $x$’s giving the maxima of $F_r(x)$ is proved to be $1/2$.

1. Introduction. Let $d(x)$ be the distance from $x$ to the nearest integer. The function $d(x)$ is continuous and periodic with period 1. Fix an integer $r \geq 2$ and define $F_r^n(x) = \sum_{k=0}^{n} d(r^k x)/r^k$. When $n$ tends to infinity, $F_r^n(x)$ converges uniformly to a continuous and periodic (with period 1) function $F_r(x)$. Further, $F_r(x)$ is proved to be everywhere nondifferentiable. As a simple example of a nowhere differentiable continuous function, T. Takagi [1] discovered $F_2(x)$ and a quarter of a century later B. L. van der Waerden [2] rediscovered $F_{10}(x)$. Takagi’s proof of the nowhere differentiability of $F_2(x)$ is applicable to any $r \geq 3$ with a slight modification when $r$ is odd. Recently, B. Martynov [3] discussed the structure of the set $E_2 = \{0 < x < 1; F_2(x) = M_2\}$ where $M_2 = \max F_2(x)$ and the result is that $x = 0.x_1x_2\cdots x_n\cdots$ (the base-4 expansion of $x$) belongs to $E_2$ if and only if $x_n = 1$ or 2 for any $n \geq 1$. From this result we can easily see that the Hausdorff dimension of $E_2$ is equal to $\log 2/\log 4 = 1/2$. This has a relation to the fact that the Hausdorff dimension of the set of zeros of the Brownian motion $B(t, \omega)$ is equal to $1/2$ and the sample functions of $B(t, \omega)$ are nowhere differentiable continuous ones for almost all $\omega$. In this paper we show that for any even $r \geq 2$ the Hausdorff dimension of the set $E_r = \{0 \leq x \leq 1; F_r(x) = M_r \equiv \max F_r(x)\}$ is equal to $1/2$ generalizing Martynov’s arguments to $r \geq 2$.

2. Functional equations.

PROPOSITION 1. The function $F_r(x)$ satisfies the following functional equations:

(1) $F_r(rx) = rF_r(x) - rd(x),$

(2) $F_r(x) = F^1_r(x) + \frac{1}{r^2} F^2_r(r^2x).$

PROOF. First,

$F_r^n(rx) = r(d(rx)/r + \cdots + d(r^{n+1}x)/r^{n+1})$

$= r(F_r^{n+1}(x) - d(x)).$
Taking the limit of the both sides, we have (1). Next,
\[
F_r(r^2x) = rF_r(rx) - rd(rx) = r(rF_r(x) - rd(x)) - rd(rx) = r^2(F_r(x) - F_r^1(x)).
\]
This implies (2).

**PROPOSITION 2.** The function \(F_r(x)\) is the unique bounded solution of the functional equation
\[
(1') f(rx) = rf(x) - rd(x).
\]
PROOF. Substituting \(r^{k-1}x\) for \(x\) in \((1')\) and dividing both sides of the resulting equation by \(r^k\), we have
\[
\frac{f(r^kx)}{r^k} = \frac{f(r^{k-1}x)}{r^{k-1}} - \frac{d(r^{k-1}x)}{r^{k-1}}.
\]
Summing up these for \(k = 1\) to \(n\), we have
\[
\frac{f(r^nx)}{r^n} = f(x) - \sum_{k=0}^{n-1} \frac{d(r^kx)}{r^k}.
\]
Letting \(n \to \infty\) in the both sides, we obtain \(f(x) = F_r(x)\).

**REMARK.** The functional equation \((1')\) for \(r = 2\) is a special case of the functional equation studied by M. Yamaguti and M. Hata [4].

**3.** \(E_r\) and \(M_r\). Observing the graphs of the functions \(F_r^1(x), F_r^2(x), \ldots\), we can easily see that (i) if \(r\) is odd, then \(E_r = \{1/2\}\) and
\[
M_r = F_r \left( \frac{1}{2} \right) = \frac{1}{2} \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots \right) = \frac{r}{2r - 2}
\]
and (ii) if \(r\) is even, then \(F_r^1(x) = 1/2\) for all \(x\) in the interval \([(r-1)/2r, (r+1)/2r]\). Therefore, in the following, we consider only the case of even \(r\)'s.

**PROPOSITION 3.**
\[
E_r \subset \left[ \frac{r}{2r + 2}, \frac{r + 2}{2r + 2} \right] = \left[ \frac{r - 1}{2r}, \frac{r + 1}{2r} \right],
\]
\[
M_r = \frac{r^2}{2r^2 - 2} \left( = F_r \left( \frac{r}{2r + 2} \right) = F_r \left( \frac{r + 2}{2r + 2} \right) \right).
\]
PROOF. First, by the periodicity of the function \(F_r(x)\), we have
\[
F_r \left( \frac{r}{2r + 2} \right) = F_r^1 \left( \frac{r}{2r + 2} \right) + \frac{1}{r^2} F_r \left( \frac{r^3}{2r + 2} \right) = \frac{1}{2} + \frac{1}{r^2} F_r \left( \frac{r}{2r + 2} \right).
\]
Therefore \(F_r(r/(2r + 2)) = r^2/(2r^2 - 2)\) and this is equal to \(F_r((r+2)/(2r + 2))\) since the function \(F_r(x)\) is symmetric with respect to 1/2. Next, take \(x \in E_r\) with \(x \leq 1/2\). Then \(F_r(rx) = rF_r(x) - rd(x) = rF_r(x) - rx \leq M_r\), therefore we have
\[
x \geq \frac{r - 1}{r} M_r \geq \frac{r - 1}{r} F_r \left( \frac{r}{2r + 2} \right) = \frac{r}{2r + 2}.
\]
From this and by the symmetric property of \( F_r(x) \) follows (3). To obtain (4) take \( x \in \mathcal{E}_r \), then

\[
M_r = F_r(x) = F_r^1(x) + \frac{1}{r^2} F_r(r^2 x) \leq \frac{1}{2} + \frac{1}{r^2} M_r.
\]

From this follows

\[
M_r \leq \frac{1}{2(1 - 1/r^2)} = \frac{r^2}{2r^2 - 2} = F_r \left( \frac{r}{2r + 2} \right),
\]

obtaining (4).

**PROPOSITION 4.** (i) If \( x \in \mathcal{E}_r \), then the fractional part of \( r^2 x \) (\( \equiv \{r^2 x\} \)) also belongs to \( \mathcal{E}_r \).

(ii) If \( F_r(r^2 x) = M_r \) and \( x \in [(r - 1)/2r, (r + 1)/2r] \), then \( x \in \mathcal{E}_r \).

**PROOF.** (i) If \( x \in \mathcal{E}_r \), then

\[
\frac{r^2}{2(r^2 - 1)} = F_r(x) = \frac{1}{2} + \frac{1}{r^2} F_r(r^2 x).
\]

Therefore \( F_r(r^2 x) = r^2/2(r^2 - 1) \) from which follows \( \{r^2 x\} \in \mathcal{E}_r \) by the periodicity of \( F_r(x) \).

(ii) By the assumptions we have

\[
F_r(x) = \frac{1}{2} + \frac{1}{r^2} F_r(r^2 x) = \frac{1}{2} + \frac{1}{2(2r^2 - 1)} = \frac{r^2}{2(r^2 - 1)},
\]

obtaining \( x \in \mathcal{E}_r \).

**PROPOSITION 5.** Suppose \( x \in \mathcal{E}_r \) and \( k \) is a positive integer. Then we have \( (x + k)/r^2 \in \mathcal{E}_r \) if and only if \( (r^2 - r)/2 \leq k \leq (r^2 + r - 2)/2 \).

**PROOF.** If \( (x + k)/r^2 \in \mathcal{E}_r \), then

\[
\frac{r}{2r + 2} \leq \frac{x + k}{r^2} \leq \frac{r + 2}{2r + 2}.
\]

To satisfy these inequalities, it is necessary for \( k \) to be in the interval \([(r^2 - r)/2, (r^2 + r - 2)/2]\) as is easily checked. Conversely, if \( (r^2 - r)/2 \leq k \leq (r^2 + r - 2)/2 \), then for \( x \in \mathcal{E}_r \) the inequalities

\[
\frac{r - 1}{2r} \leq \frac{x + k}{r^2} \leq \frac{r + 1}{2r}
\]

hold as is easily seen and \( F_r(r^2(x + k)/r^2) = F_r(x + k) = M_r \) because of the periodicity of \( F_r(x) \). By (ii) of Proposition 4 follows \( (x + k)/r^2 \in \mathcal{E}_r \).

**REMARK.** The number of the integers \( k \) satisfying the inequalities \( (r^2 - r)/2 \leq k \leq (r^2 + r - 2)/2 \) is equal to \( r \).

4. **Base-\( r^2 \) expansion.** Put \( \alpha = (r^2 - r)/2 \) and \( \beta = (r^2 + r - 2)/2 \). Let us expand the smallest and the largest numbers of \( \mathcal{E}_r \) into base-\( r^2 \) decimals (\( r^2 \)-mals). Then

\[
\frac{r}{2r + 2} = \left( \sum_{n=1}^{\infty} \frac{1}{(r^2)^n} \right) \alpha = 0.\bar{\alpha} \alpha \cdots \alpha \cdots = 0.\bar{\alpha}
\]

and

\[
\frac{r + 2}{2r + 2} = \left( \sum_{n=1}^{\infty} \frac{1}{(r^2)^n} \right) \beta = 0.\bar{\beta} \beta \cdots \beta \cdots = 0.\bar{\beta}.
\]
Generally we have the following proposition.

**Proposition 6.** Let \( x = 0.x_1x_2 \cdots x_n \cdots \) be the base-\( r^2 \) expansion of \( x \in [0,1] \). Then, \( x \in E_r \) if and only if \( \alpha \leq x_n \leq \beta \) for any \( n \geq 1 \).

**Proof.** Since \( r/(2r+2) \) and \((r+2)/(2r+2)\) belong to \( E_r \), if integers \( x_1, x_2, \ldots, x_n \) are contained in the interval \([\alpha, \beta]\), then by applying \( n \) times (ii) of Proposition 4, we have \( 0.x_1x_2 \cdots x_n \alpha \in E_r \) and \( 0.x_1x_2 \cdots x_n \beta \in E_r \). From this follows \( 0.x_1x_2 \cdots x_n \cdots \in E_r \) by the continuity of \( F_r(x) \). Conversely, if there exists an integer \( x_n \notin [\alpha, \beta] \) in the base-\( r^2 \) expansion \( 0.x_1x_2 \cdots x_n \cdots \in E_r \), then by using \( n-1 \) times (i) of Proposition 4, we have \( 0.x_nx_{n+1} \cdots \in E_r \) and this contradicts to \( E_r \subset [0.\alpha, 0.\beta] \).

5. **Hausdorff dimension.**

**Theorem.** For any even integer \( r \geq 2 \), we have \( \dim E_r = 1/2 \).

**Proof.** For all positive integers \( n \), \( E_r \) can be covered by \( r^n \), and no fewer than \( r^n \), intervals of length \( r^{-2n} \). Therefore, for every positive real number \( s < 1 \), \( E_r \) can be covered by \( r^{n+1} \), and no fewer than \( r^n \), intervals of length \( s \), where \( n = \lceil -\log s/2 \log r \rceil \). But

\[
\lim_{s \to 0} \log r^n/(\log s) = \lim_{s \to 0} \log r^{n+1}/(\log s) = 1/2,
\]

so \( \dim E_r = 1/2 \).

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**References**


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