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ON MAXIMA OF TAKAGI-VAN DER WAERDEN FUNCTIONS

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ABSTRACT. Generalizing Takagi's function $F_2(x)$ and van der Waerden's function $F_{10}(x)$, we introduce a class of nowhere differentiable continuous functions $F_r(x)$, $r \geq 2$. Some properties of $F_r(x)$ concerning especially maxima are discussed. When r is even, the Hausdorff dimension of the set of x's giving the maxima of $F_r(x)$ is proved to be $1/2$.

1. Introduction. Let $d(x)$ be the distance from x to the nearest integer. The function $d(x)$ is continuous and periodic with period 1. Fix an integer $r \geq 2$ and define $F_r^n(x) = \sum_{k=0}^n d(r^k x)/r^k$. When n tends to infinity, $F_r^n(x)$ converges uniformly to a continuous and periodic (with period 1) function $F_r(x)$. Further, $F_r(x)$ is proved to be everywhere nondifferentiable. As a simple example of a nowhere differentiable continuous function, T. Takagi [**1**] discovered $F_2(x)$ and a quarter of a century later B. L. van der Waerden [**2**] rediscovered $F_{10}(x)$. Takagi's proof of the nowhere differentiability of $F_2(x)$ is applicable to any $r \geq 3$ with a slight modification when r is odd. Recently, B. Martynov [**3**] discussed the structure of the set $E_2 = \{0 \leq x \leq 1; F_2(x) = M_2\}$ where $M_2 = \max F_2(x)$ and the result is that $x = 0.x_1x_2\cdots x_n\cdots$ (the base-4 expansion of x) belongs to E_2 if and only if $x_n = 1$ or 2 for any $n \geq 1$. From this result we can easily see that the Hausdorff dimension of E_2 is equal to $\log 2/\log 4 = 1/2$. This has a relation to the fact that the Hausdorff dimension of the set of zeros of the Brownian motion $B(t,\omega)$ is equal to $1/2$ and the sample functions of $B(t,\omega)$ are nowhere differentiable continuous ones for almost all ω. In this paper we show that for any *even* $r \geq 2$ the Hausdorff dimension of the set $E_r = \{0 \leq x \leq 1; F_r(x) = M_r \equiv \max F_r(x)\}$ is equal to $1/2$ generalizing Martynov's arguments to $r \geq 2$.

2. Functional equations.

PROPOSITION 1. *The function $F_r(x)$ satisfies the following functional equations*:

$$F_r(rx) = rF_r(x) - rd(x), \tag{1}$$

$$F_r(x) = F_r^1(x) + \frac{1}{r^2}F_r(r^2x). \tag{2}$$

PROOF. First,

$$\begin{aligned} F_r^n(rx) &= r(d(rx)/r + \cdots + d(r^{n+1}x)/r^{n+1}) \\ &= r(F_r^{n+1}(x) - d(x)). \end{aligned}$$

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Taking the limit of the both sides, we have (1). Next,

$$\begin{aligned} F_r(r^2x) &= rF_r(rx) - rd(rx) = r(rF_r(x) - rd(x)) - rd(rx) \\ &= r^2(F_r(x) - F_r^1(x)). \end{aligned}$$

This implies (2).

PROPOSITION 2. *The function $F_r(x)$ is the unique bounded solution of the functional equation*

$$f(rx) = rf(x) - rd(x). \tag{1'}$$

PROOF. Substituting $r^{k-1}x$ for x in (1′) and dividing both sides of the resulting equation by r^k, we have

$$\frac{f(r^kx)}{r^k} = \frac{f(r^{k-1}x)}{r^{k-1}} - \frac{d(r^{k-1}x)}{r^{k-1}}.$$

Summing up these for $k = 1$ to n, we have

$$\frac{f(r^nx)}{r^n} = f(x) - \sum_{k=0}^{n-1} \frac{d(r^kx)}{r^k}.$$

Letting $n \to \infty$ in the both sides, we obtain $f(x) = F_r(x)$.

REMARK. The functional equation (1′) for $r = 2$ is a special case of the functional equation studied by M. Yamaguti and M. Hata [**4**].

3. E_r and M_r. Observing the graphs of the functions $F_r^1(x)$, $F_r^2(x), \dots$, we can easily see that (i) if r is odd, then $E_r = \{1/2\}$ and

$$M_r = F_r\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 + \frac{1}{r} + \frac{1}{r^2} + \cdots\right) = \frac{r}{2r-2}$$

and (ii) if r is even, then $F_r^1(x) = 1/2$ for all x in the interval $[(r-1)/2r, (r+1)/2r]$. Therefore, in the following, we consider only the case of even r's.

PROPOSITION 3.

$$E_r \subset \left[\frac{r}{2r+2}, \frac{r+2}{2r+2}\right] \quad \left(\subset \left[\frac{r-1}{2r}, \frac{r+1}{2r}\right]\right), \tag{3}$$

$$M_r = \frac{r^2}{2r^2-2} \quad \left(= F_r\left(\frac{r}{2r+2}\right) = F_r\left(\frac{r+2}{2r+2}\right)\right). \tag{4}$$

PROOF. First, by the periodicity of the function $F_r(x)$, we have

$$\begin{aligned} F_r\left(\frac{r}{2r+2}\right) &= F_r^1\left(\frac{r}{2r+2}\right) + \frac{1}{r^2}F_r\left(\frac{r^3}{2r+2}\right) \\ &= \frac{1}{2} + \frac{1}{r^2}F_r\left(\frac{r}{2r+2}\right). \end{aligned}$$

Therefore $F_r(r/(2r+2)) = r^2/(2r^2-2)$ and this is equal to $F_r((r+2)/(2r+2))$ since the function $F_r(x)$ is symmetric with respect to $1/2$. Next, take $x \in E_r$ with $x \le 1/2$. Then $F_r(rx) = rF_r(x) - rd(x) = rF_r(x) - rx \le M_r$, therefore we have

$$x \ge \frac{r-1}{r}M_r \ge \frac{r-1}{r}F_r\left(\frac{r}{2r+2}\right) = \frac{r}{2r+2}.$$

From this and by the symmetric property of $F_r(x)$ follows (3). To obtain (4) take $x \in E_r$, then

$$M_r = F_r(x) = F_r^1(x) + \frac{1}{r^2}F_r(r^2x) \le \frac{1}{2} + \frac{1}{r^2}M_r.$$

From this follows

$$M_r \le \frac{1}{2(1-1/r^2)} = \frac{r^2}{2r^2-2} = F_r\left(\frac{r}{2r+2}\right),$$

obtaining (4).

PROPOSITION 4. (i) *If $x \in E_r$, then the fractional part of r^2x ($\equiv \{r^2x\}$) also belongs to E_r.*

(ii) *If $F_r(r^2x) = M_r$ and $x \in [(r-1)/2r, (r+1)/2r]$, then $x \in E_r$.*

PROOF. (i) If $x \in E_r$, then

$$\frac{r^2}{2(r^2-1)} = F_r(x) = \frac{1}{2} + \frac{1}{r^2}F_r(r^2x).$$

Therefore $F_r(r^2x) = r^2/2(r^2-1)$ from which follows $\{r^2x\} \in E_r$ by the periodicity of $F_r(x)$.

(ii) By the assumptions we have

$$F_r(x) = \frac{1}{2} + \frac{1}{r^2}F_r(r^2x) = \frac{1}{2} + \frac{1}{2(r^2-1)} = \frac{r^2}{2(r^2-1)},$$

obtaining $x \in E_r$.

PROPOSITION 5. *Suppose $x \in E_r$ and k is a positive integer. Then we have $(x+k)/r^2 \in E_r$ if and only if $(r^2-r)/2 \le k \le (r^2+r-2)/2$.*

PROOF. If $(x+k)/r^2 \in E_r$, then

$$\frac{r}{2r+2} \le \frac{x+k}{r^2} \le \frac{r+2}{2r+2}.$$

To satisfy these inequalities, it is necessary for k to be in the interval $[(r^2-r)/2, (r^2+r-2)/2]$ as is easily checked. Conversely, if $(r^2-r)/2 \le k \le (r^2+r-2)/2$, then for $x \in E_r$ the inequalities

$$\frac{r-1}{2r} \le \frac{x+k}{r^2} \le \frac{r+1}{2r}$$

hold as is easily seen and $F_r(r^2(x+k)/r^2) = F_r(x+k) = M_r$ because of the periodicity of $F_r(x)$. By (ii) of Proposition 4 follows $(x+k)/r^2 \in E_r$.

REMARK. The number of the integers k satisfying the inequalities $(r^2-r)/2 \le k \le (r^2+r-2)/2$ is equal to r.

4. Base-r^2 expansion. Put $\alpha = (r^2-r)/2$ and $\beta = (r^2+r-2)/2$. Let us expand the smallest and the largest numbers of E_r into *base-r^2* decimals (r^2-mals). Then

$$\frac{r}{2r+2} = \left(\sum_{n=1}^{\infty}\frac{1}{(r^2)^n}\right)\alpha = 0.\alpha\alpha\cdots\alpha\cdots = 0.\dot{\alpha}$$

and

$$\frac{r+2}{2r+2} = \left(\sum_{n=1}^{\infty}\frac{1}{(r^2)^n}\right)\beta = 0.\beta\beta\cdots\beta\cdots = 0.\dot{\beta}.$$

Generally we have the following proposition.

PROPOSITION 6. *Let $x = 0.x_1x_2\cdots x_n\cdots$ be the base-r^2 expansion of $x \in [0,1]$. Then, $x \in E_r$ if and only if $\alpha \le x_n \le \beta$ for any $n \ge 1$.*

PROOF. Since $r/(2r+2)$ and $(r+2)/(2r+2)$ belong to E_r, if integers $x_1, x_2, \ldots, x_n$ are contained in the interval $[\alpha, \beta]$, then by applying n times (ii) of Proposition 4, we have $0.x_1x_2\cdots x_n\dot{\alpha} \in E_r$ and $0.x_1x_2\cdots x_n\dot{\beta} \in E_r$. From this follows $0.x_1x_2\cdots x_n\cdots \in E_r$ by the continuity of $F_r(x)$. Conversely, if there exists an integer $x_n \notin [\alpha, \beta]$ in the base-r^2 expansion $0.x_1x_2\cdots x_n\cdots \in E_r$, then by using $n-1$ times (i) of Proposition 4, we have $0.x_nx_{n+1}\cdots \in E_r$ and this contradicts to $E_r \subset [0.\dot{\alpha}, 0.\dot{\beta}]$.

5. Hausdorff dimension.

THEOREM. *For any even integer $r \ge 2$, we have* $\dim E_r = 1/2$.

PROOF. For all positive integers n, E_r can be covered by r^n, and no fewer than r^n, intervals of length $r^{-2}n$. Therefore, for every positive real number $s < 1$, E_r can be covered by r^{n+1}, and no fewer than r^n, intervals of length s, where $n = [-\log s/2\log r]$. But

$$\lim_{s\to 0} \log r^n/(-\log s) = \lim_{s\to 0} \log r^{n+1}/(-\log s) = 1/2,$$

so $\dim E_r = 1/2$.

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