ON MAXIMA OF TAKAGI-VAN DER WAERDEN FUNCTIONS
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ABSTRACT. Generalizing Takagi's function $F_2(x)$ and van der Waerden's function $F_{10}(x)$, we introduce a class of nowhere differentiable continuous functions $F_r(x)$, $r \geq 2$. Some properties of $F_r(x)$ concerning especially maxima are discussed. When $r$ is even, the Hausdorff dimension of the set of $x$'s giving the maxima of $F_r(x)$ is proved to be $1/2$.

1. Introduction. Let $d(x)$ be the distance from $x$ to the nearest integer. The function $d(x)$ is continuous and periodic with period 1. Fix an integer $r \geq 2$ and define $F_r^n(x) = \sum_{k=0}^{n} d(r^k x) / r^k$. When $n$ tends to infinity, $F_r^n(x)$ converges uniformly to a continuous and periodic (with period 1) function $F_r(x)$. Further, $F_r(x)$ is proved to be everywhere nondifferentiable. As a simple example of a nowhere differentiable continuous function, T. Takagi [1] discovered $F_2(x)$ and a quarter of a century later B. L. van der Waerden [2] rediscovered $F_{10}(x)$. Takagi's proof of the nowhere differentiability of $F_2(x)$ is applicable to any $r \geq 3$ with a slight modification when $r$ is odd. Recently, B. Martynov [3] discussed the structure of the set $E_2 = \{0 < x < 1; F_2(x) = M_2\}$ where $M_2 = \max F_2(x)$ and the result is that $x = 0.x_1x_2 \cdots x_n \cdots$ (the base-4 expansion of $x$) belongs to $E_2$ if and only if $x_n = 1$ or 2 for any $n \geq 1$. From this result we can easily see that the Hausdorff dimension of $E_2$ is equal to $\log 2 / \log 4 = 1/2$. This has a relation to the fact that the Hausdorff dimension of the set of zeros of the Brownian motion $B(t, \omega)$ is equal to $1/2$ and the sample functions of $B(t, \omega)$ are nowhere differentiable continuous ones for almost all $\omega$. In this paper we show that for any even $r \geq 2$ the Hausdorff dimension of the set $E_r = \{0 \leq x \leq 1; F_r(x) = M_r = \max F_r(x)\}$ is equal to $1/2$ generalizing Martynov's arguments to $r \geq 2$.

2. Functional equations.

PROPOSITION 1. The function $F_r(x)$ satisfies the following functional equations:

1. \[ F_r(rx) = rF_r(x) - rd(x), \]

2. \[ F_r(x) = F_r^1(x) + \frac{1}{r^2} F_r(r^2 x). \]

PROOF. First,

\[ F_r^n(rx) = r(d(rx)/r + \cdots + d(r^{n+1} x)/r^{n+1}) \]
\[ = r(F_r^{n+1}(x) - d(x)). \]
Taking the limit of the both sides, we have (1). Next,
\[ F_r(r^2x) = rF_r(rx) - rd(rx) = r(rF_r(x) - rd(x)) - rd(rx) = r^2(F_r(x) - F_r^1(x)). \]
This implies (2).

**PROPOSITION 2.** The function \( F_r(x) \) is the unique bounded solution of the functional equation
\[(1') \quad f(rx) = rf(x) - rd(x). \]

**PROOF.** Substituting \( r^{k-1}x \) for \( x \) in \( (1') \) and dividing both sides of the resulting equation by \( r^k \), we have
\[ \frac{f(r^kx)}{r^k} = \frac{f(r^{k-1}x)}{r^{k-1}} - \frac{d(r^{k-1}x)}{r^{k-1}}. \]
Summing up these for \( k = 1 \) to \( n \), we have
\[ \frac{f(r^nx)}{r^n} = f(x) - \sum_{k=0}^{n-1} \frac{d(r^kx)}{r^k}. \]
Letting \( n \to \infty \) in the both sides, we obtain \( f(x) = F_r(x) \).

**REMARK.** The functional equation \( (1') \) for \( r = 2 \) is a special case of the functional equation studied by M. Yamaguti and M. Hata [4].

3. \( E_r \) and \( M_r \). Observing the graphs of the functions \( F_r^1(x), F_r^2(x), \ldots \), we can easily see that (i) if \( r \) is odd, then \( E_r = \{1/2\} \) and
\[ M_r = F_r \left( \frac{1}{2} \right) = \frac{1}{2} \left( 1 + \frac{1}{r} + \frac{1}{r^2} + \cdots \right) = \frac{r}{2r - 2} \]
and (ii) if \( r \) is even, then \( F_r^1(x) = 1/2 \) for all \( x \) in the interval \([ (r-1)/2r, (r+1)/2r ] \). Therefore, in the following, we consider only the case of even \( r \)’s.

**PROPOSITION 3.**
\[(3) \quad E_r \subset \left[ \frac{r}{2r + 2}, \frac{r + 2}{2r + 2} \right] \subset \left[ \frac{r - 1}{2r}, \frac{r + 1}{2r} \right], \]
\[(4) \quad M_r = \frac{r^2}{2r^2 - 2} \left( = F_r \left( \frac{r}{2r + 2} \right) = F_r \left( \frac{r + 2}{2r + 2} \right) \right) \]

**PROOF.** First, by the periodicity of the function \( F_r(x) \), we have
\[ F_r \left( \frac{r}{2r + 2} \right) = F_r^1 \left( \frac{r}{2r + 2} \right) + \frac{1}{r^2} F_r \left( \frac{r^3}{2r + 2} \right) = \frac{1}{2} + \frac{1}{r^2} F_r \left( \frac{r}{2r + 2} \right). \]
Therefore \( F_r(r/(2r + 2)) = r^2/(2r^2 - 2) \) and this is equal to \( F_r((r + 2)/(2r + 2)) \) since the function \( F_r(x) \) is symmetric with respect to 1/2. Next, take \( x \in E_r \) with \( x \leq 1/2 \). Then \( F_r(rx) = rF_r(x) - rd(x) = rF_r(x) - rx \leq M_r \), therefore we have
\[ x \geq \frac{r - 1}{r} M_r \geq \frac{r - 1}{r} F_r \left( \frac{r}{2r + 2} \right) = \frac{r}{2r + 2}. \]
From this and by the symmetric property of $F_r(x)$ follows (3). To obtain (4) take $x \in E_r$, then

$$M_r = F_r(x) = F_r^1(x) + \frac{1}{r^2} F_r(r^2 x) \leq \frac{1}{2} + \frac{1}{r^2} M_r.$$ 

From this follows

$$M_r \leq \frac{1}{2(1 - 1/r^2)} = \frac{r^2}{2r^2 - 2} = F_r\left(\frac{r}{2r + 2}\right),$$

obtaining (4).

**Proposition 4.** (i) If $x \in E_r$, then the fractional part of $r^2 x$ ($= \{r^2 x\}$) also belongs to $E_r$.

(ii) If $F_r(r^2 x) = M_r$ and $x \in [(r - 1)/2r, (r + 1)/2r]$, then $x \in E_r$.

**Proof.** (i) If $x \in E_r$, then

$$\frac{r^2}{2(r^2 - 1)} = F_r(x) = \frac{1}{2} + \frac{1}{r^2} F_r(r^2 x).$$

Therefore $F_r(r^2 x) = r^2/2(r^2 - 1)$ from which follows $\{r^2 x\} \in E_r$ by the periodicity of $F_r(x)$.

(ii) By the assumptions we have

$$F_r(x) = \frac{1}{2} + \frac{1}{r^2} F_r(r^2 x) = \frac{1}{2} + \frac{1}{2(r^2 - 1)} = \frac{r^2}{2(r^2 - 1)},$$

obtaining $x \in E_r$.

**Proposition 5.** Suppose $x \in E_r$ and $k$ is a positive integer. Then we have $(x + k)/r^2 \in E_r$ if and only if $(r^2 - r)/2 < k \leq (r^2 + r - 2)/2$.

**Proof.** If $(x + k)/r^2 \in E_r$, then

$$\frac{r}{2r + 2} \leq \frac{x + k}{r^2} \leq \frac{r + 2}{2r + 2}.$$ 

To satisfy these inequalities, it is necessary for $k$ to be in the interval $[(r^2 - r)/2, (r^2 + r - 2)/2]$ as is easily checked. Conversely, if $(r^2 - r)/2 \leq k \leq (r^2 + r - 2)/2$, then for $x \in E_r$ the inequalities

$$\frac{r - 1}{2r} \leq \frac{x + k}{r^2} \leq \frac{r + 1}{2r}$$

hold as is easily seen and $F_r(r^2(x + k)/r^2) = F_r(x + k) = M_r$ because of the periodicity of $F_r(x)$. By (ii) of Proposition 4 follows $(x + k)/r^2 \in E_r$.

**Remark.** The number of the integers $k$ satisfying the inequalities $(r^2 - r)/2 \leq k \leq (r^2 + r - 2)/2$ is equal to $r$.

**4. Base-$r^2$ expansion.** Put $\alpha = (r^2 - r)/2$ and $\beta = (r^2 + r - 2)/2$. Let us expand the smallest and the largest numbers of $E_r$ into base-$r^2$ decimals ($r^2$-mals). Then

$$\frac{r}{2r + 2} = \left(\sum_{n=1}^{\infty} \frac{1}{(r^2)^n}\right) \alpha = 0.\overline{\alpha}\cdots\alpha\cdots = 0.\dot{\alpha}$$

and

$$\frac{r + 2}{2r + 2} = \left(\sum_{n=1}^{\infty} \frac{1}{(r^2)^n}\right) \beta = 0.\overline{\beta}\cdots\beta\cdots = 0.\dot{\beta}.$$
Generally we have the following proposition.

PROPOSITION 6. Let \( x = 0.x_1x_2 \cdots x_n \cdots \) be the base-\( r^2 \) expansion of \( x \in [0,1] \). Then, \( x \in E_r \) if and only if \( \alpha \leq x_n \leq \beta \) for any \( n \geq 1 \).

PROOF. Since \( r/(2r + 2) \) and \((r + 2)/(2r + 2)\) belong to \( E_r \), if integers \( x_1, x_2, \ldots, x_n \) are contained in the interval \([\alpha, \beta] \), then by applying \( n \) times (ii) of Proposition 4, we have \( 0.x_1x_2 \cdots x_n \alpha \in E_r \) and \( 0.x_1x_2 \cdots x_n \beta \in E_r \). From this follows \( 0.x_1x_2 \cdots x_n \cdots \in E_r \) by the continuity of \( E_r(x) \). Conversely, if there exists an integer \( x_n \notin [\alpha, \beta] \) in the base-\( r^2 \) expansion \( 0.x_1x_2 \cdots x_n \cdots \in E_r \), then by using \( n - 1 \) times (i) of Proposition 4, we have \( 0.x_nx_{n+1} \cdots \in E_r \) and this contradicts to \( E_r \subset [0.\alpha, 0.\beta] \).

5. Hausdorff dimension.

THEOREM. For any even integer \( r \geq 2 \), we have \( \dim E_r = 1/2 \).

PROOF. For all positive integers \( n \), \( E_r \) can be covered by \( r^n \), and no fewer than \( r^n \), intervals of length \( r^{-2n} \). Therefore, for every positive real number \( s < 1 \), \( E_r \) can be covered by \( r^{n+1} \), and no fewer than \( r^n \), intervals of length \( s \), where

\[
 n = \left\lceil -\log s/2\log r \right\rceil.
\]

But

\[
 \lim_{s \to 0} \log r^n/(-\log s) = \lim_{s \to 0} \log r^{n+1}/(-\log s) = 1/2,
\]

so \( \dim E_r = 1/2 \).

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REFERENCES

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