**Abstract.** It is shown that if \(g \in C^1(\mathbb{C})\) with \(\bar{\partial}g\) nonvanishing on the support of \(\mu\) and if \(P^2(\mu)\) has no bounded point evaluations, then \(\text{sp}\{P^2(\mu) + gP^2(\mu)\} = L^2(\mu)\). Similar theorems stating that in the absence of bounded point evaluations \(P^2(\mu)\) is “almost” \(L^2(\mu)\) are derived. As a consequence, to show that \(P^2(\mu) = L^2(\mu)\) in the absence of bounded point evaluations, one need only show that, for example, \(\sqrt{z - \lambda} \in P^2(\mu)\) for complex \(\lambda\)’s.

Let \(\mu\) denote a finite positive Borel measure with compact support in the complex plane. Let \(P^2(\mu)\) denote the closure in \(L^2(\mu)\) of the polynomials in \(z\). A question of interest is to determine when \(P^2(\mu) = L^2(\mu)\). If \(\mu\) is supported on the boundary of the unit circle, \(\partial D\), such a characterization has been given by a classical result of Szegö [5]: either \(P^2(\mu) = L^2(\mu)\) or else \(P^2(\mu)\) has a bounded point evaluation. \(P^2(\mu)\) has a bounded point evaluation (or b.p.e.) at \(w\) in the complex plane \(C\) whenever there exists a constant \(C\) with \(0 < C < \infty\) and \(|p(w)| < C\|p\|_{L^2(\mu)}\) for all \(p \in P^2(\mu)\). For measures \(\gamma\) absolutely continuous with respect to area Lebesgue measure \(m\), a result analogous to Szegö’s theorem has been discovered by Brennan [1–4] and Hruschev (see [4]), with the mild hypothesis that \(d\gamma/dm\) belong to \(L(\log^+ L)^2(m)\). In this note we show that if \(P^2(\mu)\) has no b.p.e.’s, then \(P^2(\mu)\) and \(L^2(\mu)\) cannot differ by “too” much. As a consequence, to show that \(P^2(\mu) = L^2(\mu)\) in the absence of bounded point evaluations, one need only show that, for example, \(\sqrt{z - \lambda} \in P^2(\mu)\) for complex \(\lambda\)’s.

Denote the support of \(\mu\) by \(K\). Let \(g\) be a continuously differentiable function on \(C\) with \(\bar{\partial}g\) nonvanishing on \(K\), where \(\bar{\partial}\) denotes the operator \(1/2(\partial x + i\partial y)\). Let \(\{g_i\}_{i \in I} \subset L^\infty(\mu)\). By \(\text{sp}\{g_iP^2(\mu): i \in I\}\) we mean the \(\{\sum_{j=1}^{K} g_{ij}p_j: i_j \in I\}\) and \(p_j \in P^2(\mu)\) for \(j = 1, 2, \ldots, K\).

**Theorem 1.** Suppose \(P^2(\mu)\) has no b.p.e.’s. Then \(\text{sp}\{P^2(\mu) + gP^2(\mu)\} = L^2(\mu)\).

**Proof.** Let \(f \in L^2(\mu)\) with \(f \perp [P^2(\mu) + gP^2(\mu)]\). Then for all \(\lambda \in \mathbb{C}\) and any polynomial \(p\)

\[
0 = \int_K \frac{p(z) - p(\lambda)}{z - \lambda} (g(z) - g(\lambda)) f(z) \, d\mu(z).
\]
So

\( p(\lambda) \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) = \int_K p(z) \left( \frac{g(z) - g(\lambda)}{z - \lambda} \right) \overline{f(z)} \, d\mu(z). \)

Without loss of generality assume that \( g \in C^1_c(\mathbb{C}) \). Thus for any

\( \frac{g(z) - g(\lambda)}{z - \lambda} f(z) \in L^2(\mu). \)

Suppose that for some \( \lambda \) in \( \mathbb{C} \)

\( \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) \neq 0. \)

Then from (1) \( P^2(\mu) \) has a b.p.e. at \( \lambda \), contrary to hypothesis. Hence

\( 0 = \int_K \frac{g(z) - g(\lambda)}{z - \lambda} \overline{f(z)} \, d\mu(z) \) for all \( \lambda \) in \( \mathbb{C} \).

But by Lemma 3 in [6], for any \( h \in C^2_c(\mathbb{C}) \) with \( h \equiv 0 \) in a neighborhood of the zero set of \( \overline{\partial} g \) we have

\( h(w) = \frac{1}{\pi} \int_C \overline{\partial} s(z) \frac{g(z) - g(w)}{z - w} \, dm(z) \)

for all \( w \) in \( \mathbb{C} \) and some \( s \in C^1_c(\mathbb{C}) \). Since \( \overline{\partial} g \) does not vanish on \( K \) such \( h \)'s are dense in \( L^2(\mu) \). Combining (2) and (3) with Fubini's theorem gives \( f = 0 \) in \( L^2(\mu) \). \( \square \)

Note that it is easy to see that \( P^2(\mu) \oplus gP^2(\mu) = L^2(\mu) \) can happen only in trivial cases.

If it could be shown that \( g \) itself is in \( P^2(\mu) \) when \( P^2(\mu) \) has no b.p.e.'s, then \( P^2(\mu) = L^2(\mu) \) and the main problem is solved. This direct approach seems unlikely since the \( g \)'s for which the theorem holds are far from analytic on \( K \) and thus difficult to place in \( P^2(\mu) \). Perhaps the following version might be more useful. Here \( g \) is replaced by a collection of functions, but each function is analytic except on negligible sets with respect to \( \mu \).

For \( z = r e^{i\theta} \) with \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \), let \( \sqrt{z} = r^{1/2} e^{i\theta/2} \). Then for \( \lambda \) in \( \mathbb{C} \), \( \sqrt{z - \lambda} \) is analytic in \( z \) on \( \mathbb{C} - \{\lambda + t: t \geq 0\} \). It is easy to check that \( (z, \lambda) \mapsto \sqrt{z - \lambda} \) is Borel measurable from \( \mathbb{C} \times \mathbb{C} \to \mathbb{C} \). We have the following theorem.

**THEOREM 2.** Suppose that \( P^2(\mu) \) has no b.p.e.'s. Then

\[ \text{sp}\left\{ P^2(\mu) + \sqrt{z - \lambda} P^2(\mu): \lambda \in \mathbb{C} \right\} = L^2(\mu). \]

**PROOF.** Suppose that \( f \in L^2(\mu) \) and \( f \perp \left[ P^2(\mu) + \sqrt{z - \lambda} P^2(\mu) \right] \) for \( m \)-a.e. \( \lambda \) in \( \hat{K} \), the union of \( K \) and all the “holes” in \( K \). For \( p \) a polynomial

\[ 0 = \int_K p(z) - p(\lambda) \frac{\sqrt{z - \lambda}}{\sqrt{z - \lambda}} \overline{f(z)} \, d\mu(z) \quad \lambda \text{ a.e.-}m \text{ in } \hat{K}. \]

As before we claim that \( P^2(\mu) \) has a b.p.e. unless \( m \)-a.e. \( \lambda \in \hat{K} \) satisfies

\[ 0 = \int_K \frac{\sqrt{z - \lambda}}{\sqrt{z - \lambda}} \overline{f(z)} \, d\mu(z) = \mu_0(\lambda). \]
This follows from (4) since
\[ p(\lambda)\mu_0(\lambda) = \int_K p(z) \left[ \frac{\sqrt{z^2 - 4\lambda}}{z - \lambda} \right] d\mu(z), \]
and \( (\sqrt{z^2 - 4\lambda}/(z - \lambda))f(z) \in L^2(\mu) \) for \( m\text{-a.e.} \lambda \) in \( K \). The last fact holds by Fubini's theorem, since
\[ \int_K \left( \int_L \left| \frac{\sqrt{z - \lambda}}{z - \lambda} f(z) \right|^2 d\mu(z) \right)^{1/2} dm(\lambda) \leq C_L \|f\|_{L^2(\mu)}^2, \]
where \( L \) is a disc containing \( K \) and \( C_L \) is a constant.

We show that (5) implies that \( f = 0 \) in \( L^2(\mu) \). Let \( \phi \in C_0^\infty(L) \). Then
\[
0 = \int_L \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} f(z) d\mu(z) dm(\lambda)
= \int_K f(z) \left[ \int_L \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda) \right] d\mu(z).
\]
Let \( L_e \) denote \( L \) with an \( e \)-strip, \( S_e \), about the ray \( t + i \text{Im} z, t \leq \text{Re} z \), and the disc \( \Delta_e(z) \) removed. Then
\[ \lim_{e \downarrow 0} \int_{L_e} \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda) = \int_{L} \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda). \]

On the other hand, by Green's Theorem
\[ \int_{L_e} \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda) = \frac{1}{2i} \int_{\partial L \cap \partial L_e} \frac{\phi(\lambda)}{\sqrt{z - \lambda}} d\lambda - \int_{-\pi/2}^{\pi/2} \phi(z + ee^{i\theta}) ee^{i\theta} d\theta \]
\[ - \int_{-\infty}^0 \frac{\phi(t - i\epsilon + z)}{\sqrt{-t + i\epsilon}} dt + \int_{-\infty}^0 \frac{\phi(t + i\epsilon + z)}{\sqrt{-t - i\epsilon}} dt. \]

The first integral is 0, since \( \phi \in C_0^\infty(L) \). As \( e \downarrow 0 \) the second integral converges to 0, while the square roots in the third and fourth integrands converge to \( \sqrt{-t} \) and \( -\sqrt{-t} \), respectively. Thus
\[ \int_{L} \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda) = i \int_{-\infty}^0 \frac{\phi(t + z)}{\sqrt{-t}} dt. \]
Let \( \phi(x, y) = \psi(x)\alpha(y) \), where \( \psi \) and \( \alpha \) are in \( C_0^\infty(R) \). Then
\[ \int L \frac{\partial \phi(\lambda)}{\sqrt{z - \lambda}} dm(\lambda) = i \int \frac{\psi(x - t)}{\sqrt{t}} dt. \]

Fix \( A > \max\{|\text{Re} z|: z \in L\} \). Notice that
\[ \int \psi(x - t) x^{(n-1)}(x - t)^{n-1} dt = \int \psi(x - t)^n dt \quad \text{for } |x| \leq A. \]
This last expression is \( x + A \), \( n \)-times a polynomial in \( x \) of degree \( n \), \( p_n(x) \). (The leading coefficient of \( p_n(x) \) is \( n! u^{-1/2}(1 - u)^n du > 0 \).) Choose \( \psi \)'s to approximate \( t^n x^{(n-1)}(x - t)^{n-1} \) pointwise boundedly. Combining (6), (7), and (8), we see that \( f \perp \alpha(y)\sqrt{x + A} p_n(x) \) (since the support of \( \mu \) is contained in \( L \)). Thus by the Stone-Weierstrass theorem \( \sqrt{x + A} f = 0 \) in \( L^2(\mu) \), so \( f = 0 \) in \( L^2(\mu) \). \( \square \)
It is not difficult to replace the branch chosen for Theorem 2 by a more complicated one; say a Jordan arc which is piecewise smooth and rectifiable in $L$. In fact the choice of branch may depend on $\lambda$ if the perturbation is smooth. Also $\sqrt{z - \lambda}$ may be replaced by $\sqrt[\alpha]{z - \lambda}$, $\alpha > 0$, or, for example, by $(z - \lambda)\log(z - \lambda)$. Since the total variation of $\mu$ is finite, the $\mu$ measure of horizontal lines is zero, except for at most a countable set. Thus the $\sqrt{z - \lambda}$'s needed for Theorem 2 may be restricted so that $\sqrt{z - \lambda}$ is analytic $\mu$-a.e.

We prove a similar theorem where the branches of $\sqrt{\cdot}$ vary, but where the base of $\sqrt{\cdot}$ is fixed. For each $\alpha \in [0, 2\pi)$ we define a function $f_\alpha(z) = r^{1/2}e^{i\theta/2}$, where $z = re^{i\theta}$ is chosen so that $\alpha < \theta < 2\pi + \alpha$. For this theorem no assumption concerning b.p.e.'s is needed.

**Theorem 3.** $\text{sp}\{ P^2(\mu) + f_\alpha(z)P^2(\mu) : \alpha \in [0, 2\pi) \} = L^2(\mu)$.

**Proof.** Let $0 < \alpha < \beta < 2\pi$. Then

$$
\frac{f_\beta(z) - f_\alpha(z)}{2} = \begin{cases} 
0 & \text{if } \arg z \in [\beta, 2\pi + \alpha) \mod 2\pi, \\
-f_\alpha(z) & \text{if } \arg z \in [\alpha, \beta) \mod 2\pi.
\end{cases}
$$

Denote by $\mathcal{L}$ the closed linear span of $\{ P^2(\mu) + f_\alpha(z)P^2(\mu) : \alpha \in [0, 2\pi) \}$. From (9) $f_\alpha(z)(z: \arg z \in [\alpha, \beta) \mod 2\pi)(p(z) \in \mathcal{L}$ for every polynomial $p$ and $0 < \alpha < \beta < 2\pi$. Thus by approximation $f_\alpha(z)h(\arg z)z^n \in \mathcal{L}$ for every nonnegative integer $n$ and every continuous function $h$ on $[0, 2\pi]$ with $h(0) = h(2\pi)$. If $m$ is a nonnegative integer take $h(\arg z) = z^m/z^{|m|+n}$; if $m$ is a negative integer let $h(\arg z) = z^{-m}/|z|^{-m}$. In either case

$$
f_\alpha(z)h(\arg z)z^n = f_\alpha(re^{i\theta})r^ne^{i\theta} \in \mathcal{L},
$$

where $z = re^{i\theta}$. By the Stone-Weierstrass theorem $\text{sp}\{r^ne^{im\theta} : n \in N, m \in Z\}$ is dense in $C(K)$. Thus if $k \in L^2(\mu)$ satisfies $k \perp \mathcal{L}$, then $f_\alpha(z)k(z) \, d\mu$ is the zero measure. Hence $kd\mu = c\delta_0$, but $1 \in \mathcal{L}$ so $k = 0$ in $L^2(\mu)$. □

It is clear that only a dense subset of $\alpha$'s in $[0, 2\pi)$ is needed for Theorem 3. Again a similar argument holds for $z\log_a z$, where $\log_a(z) = \ln(z) + i \arg z$ and $\alpha \leq \arg z < \alpha + 2\pi$. Also a smooth one parameter family of nonintersecting smooth Jordan arcs emanating from a base point to $\infty$ can replace the radial lines.

**Note.** (a) The case $g(z) = \bar{z}$ of Theorem 1 was independently discovered and orally communicated to us by J. Thomson and R. Olin.

(b) When $K$ is simply connected with empty interior, arguments involving b.p.e.'s and a result analogous to Theorem 2 lead to a new proof of a theorem of Lavrentieff on the uniform approximation of continuous functions on $K$ by polynomials.

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**References**


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