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ON THE DIMENSION OF LIMITS OF INVERSE SYSTEMS

YUKINOBU YAJIMA

ABSTRACT. We say that the limit of an inverse system $X = \varprojlim\{X_\lambda, \pi^\lambda_\mu, \Lambda\}$ is *cylindrical* if each finite cozero cover of X has a σ-locally finite refinement consisting of sets of the form $\pi_\lambda^{-1}(U)$, where U is a cozero-set in X_λ and $\pi_\lambda\colon X \to X_\lambda$ is the projection.

We prove that if X is cylindrical, then $\dim X = \sup\{\dim X_\lambda\colon \lambda \in \Lambda\}$.

1. Introduction. We give a sufficient condition for the limit of an inverse system of topological spaces with the covering dimension $\leq n$ to have the dimension $\leq n$.

We say that the limit of an inverse system $X = \varprojlim\{X_\lambda, \pi^\lambda_\mu, \Lambda\}$ is cylindrical if each finite cozero cover of X has a σ-locally finite refinement consisting of sets of the form $\pi_\lambda^{-1}(U)$, where U is a cozero-set in X_λ and $\pi_\lambda\colon X \to X_\lambda$ is the projection. In this paper, we prove

"If X is cylindrical, then $\dim X = \sup\{\dim X_\lambda\colon \lambda \in \Lambda\}$".

Next, we show that the limit of an inverse system of paracompact spaces and perfect maps is cylindrical and that the limit of an perforable inverse sequence of normal spaces is cylindrical, thus obtaining earlier results of Katuta-Pasynkov [**K, P₁**] and Nagami-Pasynkov [**N₂, P₂**], respectively, as corollaries. Since every Cartesian product of infinitely many spaces is the limit of the inverse system of its finite subproducts, we immediately obtain a sufficient condition for the dimension of the Cartesian product not to exceed the supremum of the dimensions of its finite subproducts. In particular, this condition is satisfied for a Cartesian product of metrizable spaces, thus yielding E. Pol's result [**Po**] as a corollary.

2. Preliminaries. Throughout this paper, by a space and a map we mean a topological space and a continuous map, respectively. No separation axioms are assumed. However, regular, normal and paracompact spaces are always assumed to be T_2. For a set A, by $|A|$ we mean the cardinality of A. Natural numbers are denoted by i, j, k and m. The smallest infinite ordinal (cardinal) number is denoted by ω ($\aleph_0$).

The dimension dim always means the covering dimension, for which we use a nonnegative integer n. For a space X, a *cozero-set* in X is a set of the form $f^{-1}((0,1])$ for some map $f\colon X \to [0,1]$.

Let $\{X_\lambda, \pi^\lambda_\mu, \Lambda\}$ be an inverse system with the limit X. Then a map $\pi_\lambda\colon X \to X_\lambda$ means the projection for each $\lambda \in \Lambda$. The directed set Λ is omitted if $|\Lambda| \leq \aleph_0$.

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Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a Cartesian product. For each $\xi \subset \Lambda$, let $X_\xi = \prod_{\lambda \in \xi} X_\lambda$ and $\pi_\xi \colon X \to X_\xi$ the projection. For each $\xi \subset \Lambda$ ($\Gamma \subset \Lambda$) with $|\xi| < \aleph_0$ ($|\Gamma| \leq \aleph_0$), the product X_ξ (X_Γ) is said to be a *finite* (*countable*) *subproduct* of X.

3. Main theorem. The following definition is similar to that of rectangularity for finite products in the sense of Pasynkov [$\mathbf{P_3}$].

DEFINITION 1. The limit of an inverse system $X = \varprojlim\{X_\lambda, \pi_\mu^\lambda, \Lambda\}$ is said to be *cylindrical*, if each finite cozero cover of X has a σ-locally finite refinement by sets of the form $\pi_\lambda^{-1}(U)$, where $\lambda \in \Lambda$ and U is a cozero-set in X_λ. Such a set $\pi_\lambda^{-1}(U)$ is said to be a *cozero cylinder* in X.

Our main theorem is as follows.

THEOREM. *Let* $X = \varprojlim\{X_\lambda, \pi_\mu^\lambda, \Lambda\}$ *be the limit of an inverse system of spaces* X_λ *with* $\dim X_\lambda \leq n$. *If* X *is cylindrical, then* $\dim X \leq n$.

The idea of the proof of Theorem is essentially due to Pasynkov [$\mathbf{P_3}$, Theorem 17]. Indeed, we make use of the following result in [$\mathbf{P_3}$, Proposition 10].

LEMMA 1. *Suppose that for each finite cozero cover* $\mathcal{O}$ *of a space* X *there exists an inverse system* $\{R_\alpha, \phi_\beta^\alpha, A\}$ *of metrizable spaces* R_α *with* $\dim R_\alpha \leq n$, *satisfying the following*:

(a) *for each* $\alpha \in A$, *there exists a map* $f_\alpha \colon X \to R_\alpha$ *such that* $f_\beta = \phi_\beta^\alpha \circ f_\alpha$ *for* $\beta < \alpha$,

(b) *there exist some* $A_0 \subset A$ *and open sets* V_α *in* R_α, $\alpha \in A_0$, *such that* $\{f_\alpha^{-1}(V_\alpha) \colon \alpha \in A_0\}$ *is a* σ*-locally finite refinement of* $\mathcal{O}$.

Then we have $\dim X \leq n$.

PROOF OF THEOREM. Let $\mathcal{O}$ be any finite cozero cover of X. By the assumption, it has a σ-locally finite refinement $\{\pi_{\lambda_\alpha}^{-1}(U_\alpha) \colon \alpha \in A\}$ by cozero cylinders in X. For each $\alpha \in A$, we take a map $g_\alpha \colon X_{\lambda_\alpha} \to [0,1]$ such that $g_\alpha^{-1}((0,1]) = U_\alpha$. By Pasynkov's factorization theorem (cf. [$\mathbf{P_3}$, Theorem 2]), there exist a metrizable space R_α with $\dim R_\alpha \leq n$, an "onto" map $f_\alpha \colon X_{\lambda_\alpha} \to R_\alpha$ and a map $h_\alpha \colon R_\alpha \to [0,1]$ such that $g_\alpha = h_\alpha \circ f_\alpha$ for each $\alpha \in A$. Put $V_\alpha = h_\alpha^{-1}((0,1])$ and $\tilde{f}_\alpha = f_\alpha \circ \pi_{\lambda_\alpha}$ for each $\alpha \in A$. Then we have

(1) $\tilde{f}_\alpha^{-1}(V_\alpha) = \pi_{\lambda_\alpha}^{-1}(U_\alpha)$.

Let A^* be the set of all finite subsets of A. For each $a \in A^*$, we construct a metrizable space R_a with $\dim R_a \leq n$, a $\lambda(a) \in \Lambda$ with $\lambda(a) \geq \lambda(b)$ for $b \subset a$, an "onto" map $f_a \colon X_{\lambda(a)} \to R_a$ and maps $\phi_b^a \colon R_a \to R_b$ for $b \subset a$, satisfying the following:

(2) $R_a = R_\alpha$, $\lambda(a) = \lambda_\alpha$ and $f_a = f_\alpha$ if $a = (\alpha) \in A^*$,

(3) $\tilde{f}_b = \phi_b^a \circ \tilde{f}_a$ for $b \subset a$, where $\tilde{f}_a = f_a \circ \pi_{\lambda(a)}$,

(4) $\phi_c^a = \phi_c^b \circ \phi_b^a$ for $c \subset b \subset a$.

For each $a \in A^*$ with $|a| = 1$, we define R_a, $\lambda(a)$ and f_a as in (2). Assume that the above construction has been already performed for each $a \in A^*$ with $|a| < k$. Consider an $a \in A^*$ with $|a| = k$. We can choose a $\lambda(a) \in \Lambda$ with $\lambda(a) \geq \lambda(b)$ for each $b \subsetneq a$. We put $S_a = \prod_{b \subsetneq a} R_b$. We take the map $g_a \colon X_{\lambda(a)} \to S_a$ defined by $g_a(x) = (f_b \circ \pi_{\lambda(b)}^{\lambda(a)}(x))_{b \subsetneq a}$ for each $x \in X_{\lambda(a)}$. Again, by Pasynkov's factorization theorem, there exist a metrizable space R_a with $\dim R_a \leq n$, an "onto" map $f_a \colon X_{\lambda(a)} \to R_a$ and a map $h_a \colon R_a \to S_a$ such that $g_a = h_a \circ f_a$. Let

$b \subsetneq a$. Let $p_b^a\colon S_a \to R_b$ be the projection. We define the map $\phi_b^a\colon R_a \to R_b$ as $\phi_b^a = p_b^a \circ h_a$. Then we have

(5) $\phi_b^a \circ f_a = f_b \circ \pi_{\lambda(b)}^{\lambda(a)}$.

By (5), (3) is satisfied. Moreover, by (5), we obtain $\phi_c^b \circ \phi_b^a \circ f_a = \phi_c^a \circ f_a$ for each $c \subset b$. Since f_a is "onto", (4) is satisfied. Thus, we have inductively accomplished the desired construction.

By (4), $\{R_a, \phi_b^a, A^*\}$ is an inverse system. For each $a \in A^*$, R_a, $\tilde{f}_a$ and ϕ_b^a satisfy (a) of Lemma 1. Put $A_0 = \{a \in A^*\colon\ |a| = 1\}$. Moreover, let $V_a = V_\alpha$ for each $a = (\alpha) \in A_0$. By (1) and (2), we have

$$\{\tilde{f}_a^{-1}(V_a)\colon\ a \in A_0\} = \{\pi_{\lambda_\alpha}^{-1}(U_\alpha)\colon\ \alpha \in A\}.$$

Hence (b) of Lemma 1 is satisfied. Thus all the conditions of Lemma 1 are satisfied, which concludes $\dim X \le n$. The proof is complete.

4. Cylindrical limits of inverse systems. Here, we show two propositions which give different kinds of cylindrical limits of inverse systems. Our theorem and these propositions immediately yield two earlier results as corollaries. First, we show

PROPOSITION 1. *Let $X = \varprojlim\{X_\lambda, \pi_\mu^\lambda, \Lambda\}$ be the limit of an inverse system of paracompact spaces and perfect maps. Then it is cylindrical.*

PROOF. We can assume that each π_λ and π_μ^λ is "onto". Since each π_μ^λ is perfect, so is each π_λ. Pick a $\lambda_0 \in \Lambda$ and fix it. Let $\mathcal{U}$ be any open cover of X. We can assume $\mathcal{U} = \{\pi_{\lambda_\alpha}^{-1}(U_\alpha)\colon\ \alpha \in A\}$, where U_α is a cozero-set in X_{λ_α} for each $\alpha \in A$. Pick an $x \in X_{\lambda_0}$. Since $\pi_{\lambda_0}^{-1}(x)$ is compact, we can choose a finite subset $A(x)$ of A such that $\pi_{\lambda_0}^{-1}(x) \subset \bigcup_{\alpha \in A(x)} \pi_{\lambda_\alpha}^{-1}(U_\alpha)$. Since π_{λ_0} is closed, we can take up an open neighborhood W_x of x in X_{λ_0} such that

$$\pi_{\lambda_0}^{-1}(x) \subset \pi_{\lambda_0}^{-1}(W_x) \subset \bigcup_{\alpha \in A(x)} \pi_{\lambda_\alpha}^{-1}(U_\alpha).$$

Since X_{λ_0} is paracompact, there exists a locally finite cozero refinement $\mathcal{G}$ of $\{W_x\colon\ x \in X_{\lambda_0}\}$. Pick a $G \in \mathcal{G}$. Take $p_G \in X_{\lambda_0}$ with $G \subset W_{p_G}$. We choose a $\lambda_G \in \Lambda$ with $\lambda_G \ge \sup\{\lambda_\alpha\colon\ \alpha \in A(p_G)\}$ and $\lambda_G \ge \lambda_0$. Let

$$\mathcal{V}(G) = \{(\pi_{\lambda_0}^{\lambda_G})^{-1}(G) \cap (\pi_{\lambda_\alpha}^{\lambda_G})^{-1}(U_\alpha)\colon\ \alpha \in A(p_G)\}.$$

Then we can verify that $\{\pi_{\lambda_G}^{-1}(V)\colon\ V \in \mathcal{V}(G) \text{ and } G \in \mathcal{G}\}$ is a locally finite refinement of $\mathcal{U}$ by cozero cylinders in X. Hence X is cylindrical. The proof is complete.

REMARK 1. Pasynkov [**P₁**] announced a similar result to Proposition 1 without the proof.

COROLLARY 1 [**K, P₁**]. *Let $X = \varprojlim\{X_\lambda, \pi_\mu^\lambda, \Lambda\}$ be the limit of an inverse system of paracompact spaces X_λ such that* $\dim X_\lambda \le n$ *and perfect maps π_μ^λ. Then* $\dim X \le n$.

For the next proposition, we state the following concept which has been introduced earlier in connection with the dimension of the limits of inverse sequences.

DEFINITION 2 [$\mathbf{N_2}$, $\mathbf{P_2}$]. An inverse sequence $\{X_i, \pi^i_j\}$ is said to be *perforable* if for any sequence $\{O_i\}_{i<\omega}$ of open sets O_i in X_i with $(\pi^i_j)^{-1}(O_j) \subset O_i$, $i \geq j$, and $\bigcup_{i<\omega} \pi_i^{-1}(O_i) = X$, there exists a sequence $\{F_i\}_{i<\omega}$ of closed sets F_i in X_i such that $F_i \subset O_i$ for each $i < \omega$ and $\bigcup_{i<\omega} \pi_i^{-1}(F_i) = X$.

The inverse sequences with the perforable property have been enumerated in [$\mathbf{N_2}$, Remark 4.2] and [$\mathbf{P_2}$, Corollary 1].

PROPOSITION 2. *If an inverse sequence $\{X_i, \pi^i_j\}$ of normal spaces X_i is perforable, then the limit X is cylindrical.*

PROOF. Let $\{G_k\}_{k\leq m}$ be any finite cozero cover of X. For each $i < \omega$ and $k \leq m$, let U_i^k be the maximal open set in X_i with $\pi_i^{-1}(U_i^k) \subset G_k$. Put $O_i = \bigcup_{k\leq m} U_i^k$. Then the sequence $\{O_i\}_{i<\omega}$ satisfies the conditions of Definition 2. By the assumption, there exists a sequence $\{F_i\}_{i<\omega}$ of closed sets F_i in X_i, described in Definition 2. Since each X_i is normal, there exists a sequence $\{C_i^k\colon\ i < \omega$ and $k \leq m\}$ of cozero-sets C_i^k in X_i such that $C_i^k \subset U_i^k$ and $F_i \subset \bigcup_{k\leq m} C_i^k$ for each $i < \omega$ and $k \leq m$. Then $\{\pi_i^{-1}(C_i^k)\colon\ i < \omega$ and $k \leq m\}$ is a countable refinement of $\{G_k\}_{k\leq m}$ by cozero cylinders in X. Hence X is cylindrical. The proof is complete.

COROLLARY 2 [$\mathbf{N_2}$, $\mathbf{P_2}$]. *Let $X = \varprojlim\{X_i, \pi^i_j\}$ be the limit of an inverse sequence of normal spaces X_i with $\dim X_i \leq n$. If the sequence is perforable, then $\dim X \leq n$.*

5. Cylindrical Cartesian products. Here, each Cartesian product is an infinite one unless otherwise stated. Note that a Cartesian product X is represented as the limit of the inverse system of all finite subproducts of X and their projections. So we adopt the following definition for convenience.

DEFINITION 3. A Cartesian product $X = \prod_{\lambda\in\Lambda} X_\lambda$ is said to be *cylindrical* if each finite cozero cover of X has a σ-locally finite refinement by sets of the form $\pi_\xi^{-1}(U)$, where $\xi \subset \Lambda$ with $|\xi| < \aleph_0$ and U is a cozero-set in X_ξ. Moreover, such a set $\pi_\xi^{-1}(U)$ is said to be a *cozero cylinder* in X.

By the Theorem, we immediately obtain

COROLLARY 3. *If a Cartesian product $X = \prod_{\lambda\in\Lambda} X_\lambda$ is cylindrical, then*

$$\dim X = \sup\{\dim X_\xi\colon\ \xi \subset \Lambda \text{ with } |\xi| < \aleph_0\}.$$

Now, we consider when a Cartesian product is cylindrical. We prepare two lemmas concerning Cartesian products of paracompact Σ-spaces.

LEMMA 2. *Let X_i be a paracompact Σ-space for each $i < \omega$. Then each open cover of $\prod_{i<\omega} X_i$ has a σ-locally finite refinement by cozero cylinders.*

Lemma 2 can be obtained by a modification of the proofs from [$\mathbf{N_1}$, Theorems 3.6 and 3.13]. Moreover, Lemma 2 coupled with the proof of [$\mathbf{Y}$, Theorem 1] essentially yields the following result.

LEMMA 3. *Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a Cartesian product of paracompact Σ-spaces, each finite subproduct of which has countable tightness. Let Σ be a Σ-product of $\{X_\lambda\}_{\lambda\in\Lambda}$. Then, for all disjoint closed sets A and B in Σ, there exists a σ-locally* *finite cover $\mathcal{U}$ of X by cozero cylinders in X such that each member of $\mathcal{U}$ is disjoint from A or B.*

PROPOSITION 3. *A Cartesian product of paracompact Σ-spaces, each finite subproduct of which has countable tightness, is cylindrical.*

PROOF. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a Cartesian product as the above and Σ a Σ-product of $\{X_\lambda\}_{\lambda\in\Lambda}$. Let G be any cozero-set in X. We take a sequence $\{Z_i\}_{i<\omega}$ of zero-sets in X such that $G = \bigcup_{i<\omega} Z_i$. Fix an $i < \omega$. By Lemma 3, there exists a σ-locally finite cover $\mathcal{U}_i$ of X by cozero cylinders in X such that each $U \in \mathcal{U}_i$ is disjoint from $\Sigma \setminus G$ or $Z_i \cap \Sigma$. Since $X \setminus G$ and Z_i are closed G_δ-sets in X, by [**PP**, Lemma 2], we have

$$X \setminus G = \mathrm{Cl}(\Sigma \setminus G) \quad \text{and} \quad Z_i = \mathrm{Cl}(Z_i \cap \Sigma),$$

where Cl denotes the closure in X. We set

$$\mathcal{U}_i^* = \{U \in \mathcal{U}_i \colon U \cap (\Sigma \setminus G) = \emptyset\}.$$

Then each $U \in \mathcal{U}_i^*$ and $U \in \mathcal{U}_i \setminus \mathcal{U}_i^*$ are disjoint from $X \setminus G$ and Z_i, respectively. Since $\mathcal{U}_i$ is a cover of X, we have $Z_i \subset \bigcup\{U\colon\ U \in \mathcal{U}_i^*\} \subset G$. So we obtain $G = \bigcup\{U\colon\ U \in \mathcal{U}_i^* \text{ and } i < \omega\}$. Thus G is a σ-locally finite union of cozero cylinders in X. Hence X is cylindrical. The proof is complete.

REMARK 2. For a Cartesian product X as in Proposition 3, it follows from [**PP**, Proposition 2], [**Tk**, Theorem 1] and [**Y**, Theorem 1] that each open F_σ-set in X is a cozero-set. So, Proposition 3 is a generalization of [**K1**, Theorem 1].

PROPOSITION 4. *Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a Cartesian product of regular spaces. Then X is cylindrical if one of the following properties is true:*

(a) *Each countable subproduct of X is Lindelöf.*

(b) *Each finite subproduct of X is hereditarily Lindelöf.*

PROOF. First, assume that each finite subproduct of X is Lindelöf. Let G be any cozero-set in X. We take a map $g\colon X \to [0,1]$ such that $g^{-1}((0,1]) = G$. By [**E**, Theorem 1], there exist a $\Gamma \subset \Lambda$ with $|\Gamma| \le \aleph_0$ and a map $f\colon\ X_\Gamma \to [0,1]$ such that $g = f \circ \pi_\Gamma$. Let $U = f^{-1}((0,1])$.

Case of (a). Since U is an F_σ-set of a Lindelöf space X_Γ, so is U. Hence we can choose a sequence $\{V_i\}_{i<\omega}$ of cozero cylinders in X_Γ such that $U = \bigcup_{i<\omega} V_i$. Then $H_i = \pi_\Gamma^{-1}(V_i)$ is a cozero cylinder in X for each $i < \omega$. One can easily verify $G = \bigcup_{i<\omega} H_i$.

Case of (b). Let $\Gamma = \{\lambda_1, \lambda_2, \ldots\}$ and $\xi_i = \{\lambda_1, \ldots, \lambda_i\}$ for each $i < \omega$. Fix an $i < \omega$. Let W_i be the maximal open set X_{ξ_i} such that $W_i \times \prod_{k>i} X_{\lambda_k} \subset U$. Since W_i is Lindelöf, there exists a sequence $\{W_{ij}\}_{j<\omega}$ of cozero-sets in X_{ξ_i} such that $W_i = \bigcup_{j<\omega} W_{ij}$. Then $H_{ij} = \pi_{\xi_i}^{-1}(W_{ij})$ is a cozero cylinder in X for each $j < \omega$. Here, running $i < \omega$, one can verify $G = \bigcup_{i,j<\omega} H_{ij}$.

In both cases, G is a countable union of cozero cylinders in X. Hence X is cylindrical. The proof is complete.

By the Theorem and Propositions 3 and 4, we can obtain some earlier results.

COROLLARY 4. *Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be a Cartesian product of regular spaces. Then we have*

$$\dim X = \sup\{\dim X_\xi\colon\ \xi \subset \Lambda \textit{ with } |\xi| < \aleph_0\}$$

if one of the following conditions (a)–(c) *is satisfied*:

(a) *Each* X_λ *is a paracompact* Σ*-space and each finite subproduct* X_ξ *has countable tightness* [**Y**]. *In particular, each* X_λ *is metrizable* [**Po**].

(b) *Each countable subproduct of* X *is Lindelöf* [**C, M**].

(c) *Each finite subproduct* X_ξ *is hereditarily Lindelöf. In particular, it is perfectly normal and Lindelöf* [**C**].

REMARK 3. Let S be the Sorgenfrey line. Terasawa [**T**] showed that $\dim S^\alpha = 0$ for each $\alpha \geq \omega$. Moreover, one can see from his proof that S^α is cylindrical for each $\alpha \geq \omega$.

ADDED IN PROOF. Pasynkov [Soviet Math. Dokl. **26** (1982), 654–685] has introduced the concept of piecewise rectangularity for Cartesian products, which is weaker than that of the cylindrical property. Moreover, he has announced that the equality in our Corollary 3 is true for a Cartesian product which is piecewise rectangular.

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DEPARTMENT OF MATHEMATICS, KANAGAWA UNIVERSITY, YOKOHAMA 221, JAPAN