PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 3, July 1984

PRIMITIVE OBSTRUCTIONS IN THE COHOMOLOGY OF LOOPSPACES

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ABSTRACT. Let X and X' be H-spaces. If $f\colon \Omega X \to \Omega X'$ is an H-map then the obstruction to f being a homotopy-commutative map is a subset $\{c_2(f)\} \subset [\Omega X \wedge \Omega X; \Omega^2 X']$. In this paper we prove: *If* $[f]$ *is in the image of the composition*

$$[P_{k+m}\Omega X; X'] \to [\Sigma\Omega X; X'] \overset{\approx}{\to} [\Omega X; \Omega X'],$$

then $\{c_2(f)\}$ *is in the image of the composition*

$$[P_k\Omega X \wedge P_m\Omega X; X'] \to [\Sigma\Omega X \wedge \Sigma\Omega X; X'] \overset{\approx}{\to} [\Omega X \wedge \Omega X; \Omega^2 X'].$$

Consequently if $\alpha \in H^n(\Omega X; Z_p)$ is an A_3-class in the sense of Stasheff then each element of $\{c_2(f)\}$ is of the form $\sum c_i' \otimes c_i''$ where the c_i'' are primitive.

1. The purpose of this note is to develop a decomposition formula for a certain obstruction class that occurs in the study of H-spaces. Let G and G' be associative H-spaces and $f\colon G \to G'$ be an H-map. For homotopy-commutative G and G' we introduced in [**6**] the notion of f being a C_2-map. Specifically, if q and q' are the commuting homotopies for G and G', respectively, and m is a homotopy from $f(xy)$ to $f(x)f(y)$, then f is a C_2-*map* provided that there exists a secondary homotopy $r\colon I^2 \times G^2 \to G'$ such that $r(0, t, x, y) = f(q(t, x, y))$, $r(1, t, x, y) = q'(t, f(x), f(y))$, $r(s, 0, x, y) = m(s, x, y)$, and $r(s, 1, x, y) = m(s, y, x)$. The obstruction to the existence of r is an element $c_2(f)$ of $[G \wedge G; \Omega G']$, cf. [**7**]. Different choices of m give us a set of obstructions $\{c_2(f)\} \subset [G \wedge G; \Omega G']$.

The sets $\{c_2(f)\}$ have proved to be useful in the study of H-spaces, see for example [**7, 1**]. We shall deal exclusively with the case $G = \Omega X$, $G' = \Omega X'$, where X and X' are H-spaces and q, q' are the usual commuting homotopies for the loop multiplications. Now if $G' = K(Z_p, n)$, then

$$\{c_2(f)\} \subset \left[\Omega X \wedge \Omega X; \Omega K(Z_p, n)\right] \approx H^{n-1}(\Omega X \wedge \Omega X; Z_p).$$

Zabrodsky proved in [**7**] that if $\alpha \in H^n(\Omega X, Z_p)$ is a suspension element then the elements of $\{c_2(f)\}$ may be written in the form $\sum c_i' \otimes c_i''$ where the classes c_i' and c_i'' are also suspensions. One might hope that if α were merely a primitive element then the c_i' and c_i'' might also turn out to be primitive. This is in general false. In

Received by the editors July 25, 1983.
1980 *Mathematics Subject Classification*. Primary 55P45, 55S20.
Key words and phrases. H-space, homotopy-commutativity, obstruction.

order to describe what *is* true in this case, we refer to Stasheff [3], for the definition of the *projective spaces* of G,

$$\Sigma G = P_1G \subset P_2G \subset \cdots \subset \bigcup_k P_kG = P_\infty G \sim X.$$

A map $f\colon G \to G'$ is called an *A_k-map* provided that $\Sigma f\colon \Sigma G \to \Sigma G'$ extends to a map of filtered spaces $\{P_if\}\colon \{P_iG\} \to \{P_iG'\}$. The adjoint of the inclusion $i(1, k)$: $\Sigma G \to P_kG$ is an A_k-map and the adjoint to $i(1, \infty)$ gives the homotopy equivalence of G with $\Omega P_\infty G$. Under this equivalence the A_∞-maps correspond to the loop maps. Furthermore, the A_2-maps are simply the H-maps. A class $\alpha \in [G; G']$ is represented by an A_k-map if and only if it is in the image of the composition $[P_kG; P_\infty G'] \to [\Sigma G; P_\infty G'] \stackrel{\approx}{\to} [G; G']$. We shall prove

1.1. THEOREM. *If G and G' are loop spaces of H-spaces and $f\colon G \to G'$ is an A_{k+m}-map, then $\{c_2(f)\}$ is in the image of the composition*

$$[P_kG \wedge P_mG; P_\infty G'] \to [\Sigma G \wedge \Sigma G; P_\infty G'] \stackrel{\approx}{\to} [G \wedge G; \Omega G'].$$

Specializing to the case $G' = K(Z_p, n)$, Theorem 1.1 and the Künneth formula give us

1.2. THEOREM. *Let X be an H-space. If $\alpha \in H^n(\Omega X; Z_p)$ is an A_{k+m}-class, then $\{c_2(\alpha)\}$ consists of elements of the form $c_2(\alpha) = \sum c_i' \otimes c_i''$, where the c_i' (resp. c_i'') are A_k-classes (resp. A_m-classes).*

(Since the A_∞-classes are the suspension elements, we note that the case $k = m = \infty$ is the above-mentioned result of Zabrodsky.)

In particular, we see that if α is an A_3-class then we may write elements of $\{c_2(f)\}$ in the form $c_2(f) = \sum c_i' \otimes c_i''$, where the c_i'' are primitive. The import of Theorem 1.2 is thus that the types of elements that occur in the formula for $c_2(f)$ are considerably restricted. This fact is now being applied in the further investigation of the cohomology of finite H-spaces (cf. [2]), to extend results originally proved in [5].

2. The proof of Theorem 1.1 consists of the identification of $c_2(f)$ with another obstruction, $\theta(f)$, together with some routine diagram-chasing. Let us write $G = \Omega X$, $G' = \Omega X'$, for H-spaces X and X'. The homotopy equivalence $X \sim P_\infty G$ induces a multiplication μ on $P_\infty G$ that may be taken to be filtration-preserving, i.e. $\mu(P_kG \times P_mG) \subset P_{k+m}G$ [4]. If $f\colon G \to G'$ is an A_k-map, $k \geqslant 2$, we have the (not necessarily commutative) diagram:

$$\begin{array}{ccccc} \Sigma G \times \Sigma G & \xrightarrow{\mu} & P_2G & & \\ \downarrow{\scriptstyle \Sigma f \times \Sigma f} & & \downarrow{\scriptstyle P_2 f} & & \\ \Sigma G' \times \Sigma G' & \xrightarrow{\mu'} & P_2G' & \xrightarrow{i'(2,\infty)} & P_\infty G' \end{array}$$

The obstruction $\theta(f)$ to a homotopy between $i'(2, \infty) \circ \mu' \circ (\Sigma f \times \Sigma f)$ and $i'(2, \infty) \circ P_2 f \circ \mu$ is an element of $[\Sigma G \wedge \Sigma G; P_\infty G']$. The next proposition relates $\theta(f)$ to the question at hand.

2.1. PROPOSITION. *Under the isomorphism* $[\Sigma G \wedge \Sigma G; P_\infty G] \approx [G \wedge G; \Omega G']$, $\theta(f)$ *goes to* $c_2(f)$.

PROOF. This proposition is essentially a version of the adjoint relationship between Whitehead products and Samelson products. It follows from carefully depicting the maps and homotopies that are involved. □

Theorem 1.1 now follows from

2.2. PROPOSITION. *If* $f\colon G \to G'$ *is an* A_{k+m}*-map, then* $\theta(f)$ *is in the image of*

$$(i(1,k) \wedge i(1,m))^*\colon [P_kG \wedge P_mG; P_\infty G'] \to [\Sigma G \wedge \Sigma G; P_\infty G].$$

PROOF. By definition

$$\theta(f) = i'(2,\infty)\circ\mu'\circ(\Sigma f \times \Sigma f) - i'(2,\infty)\circ P_2 f\circ\mu.$$

By the commutative diagram

$$\begin{tikzcd}
\Sigma G \times \Sigma G \arrow[r, "i(1,k)\times i(1,m)"] \arrow[d, "\Sigma f \times \Sigma f"] & P_kG \times P_mG \arrow[d, "P_kf \times P_mf"] \\
\Sigma G' \times \Sigma G' \arrow[r, "i'(1,k)\times i'(1,m)"] \arrow[d, "\mu'"] & P_kG' \times P_mG' \arrow[d, "\mu'"] \\
P_2G' \arrow[r, "i'(2,k+m)"] \arrow[dr, "i'(2,\infty)"'] & P_{k+m}G' \arrow[d] \\
 & P_\infty G'
\end{tikzcd}$$

we see that

$$i'(2,\infty)\circ\mu'\circ(\Sigma f \times \Sigma f) \sim i'(k+m,\infty)\circ\mu'\circ(P_kf \times P_mf)\circ(i(1,k)\times i(1,m)).$$

And, by the commutative diagram

$$\begin{tikzcd}
\Sigma G \times \Sigma G \arrow[r, "i(1,k)\times i(1,m)"] \arrow[d, "\mu"] & P_kG \times P_mG \arrow[d, "\mu"] \\
P_2G \arrow[r, "i(2,k+m)"] \arrow[d, "P_2f"] & P_{k+m}G \arrow[d, "P_{k+m}f"] \\
P_2G' \arrow[r, "i'(2,k+m)"] \arrow[dr, "i'(2,\infty)"'] & P_{k+m}G' \arrow[d, "i'(k+m,\infty)"] \\
 & P_\infty G'
\end{tikzcd}$$

we obtain

$$i'(2,\infty)\circ P_2f\circ\mu \sim i'(k+m)\circ P_{k+m}f\circ\mu\circ(i(1,k)\times i(1,m)).$$

So

$$\theta(f) = \left[i'(k+m,\infty)\circ\mu'\circ(P_kf \times P_mf) - i'(k+m,\infty)\circ P_{k+m}f\circ\mu\right] \circ(i(1,k)\times 1(1,m)). \quad \square$$

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