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# ON THE RATIONALITY OF THE VARIETY OF SMOOTH RATIONAL SPACE CURVES WITH FIXED DEGREE AND NORMAL BUNDLE

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ABSTRACT. Let $\tilde{S}_{n,a}$ be the variety of smooth, rational curves of degree $n$ in $\mathbf{P}_3$ whose normal bundle has a factor of degree $2n - 1 + a$ and a factor of degree $2n - 1 - a$. In this paper we prove that $\tilde{S}_{n,a}$ is rational if $n - a$ is even and $a > 0$.

We work over $\mathbf{C}$. Let $\tilde{S}_{n,a} \subset \operatorname{Hilb} \mathbf{P}_3$ be the set of smooth, rational curves in $\mathbf{P}_3$ of degree $n$ whose normal bundle splits with a summand of degree $2n - 1 - a$ and another of degree $2n - 1 + a$. Eisenbud and Van de Ven [**1, 2**] proved that for $0 \leqslant a \leqslant n - 4$, $\tilde{S}_{n,a}$ is not empty, irreducible and of dimension $4n - 2a + 1$ (if $a > 0$). Let $S_{n,a}$ be the set of embeddings $f\colon \mathbf{P}_1 \to \mathbf{P}_3$ with $f(\mathbf{P}_1) \in \tilde{S}_{n,a}$. They proved in [**2**] that $S_{n,a}$ is irreducible, rational and, if $a > 0$, of dimension $4n - 2a + 4$. $\mathrm{PGL}(2) = \operatorname{Aut}(\mathbf{P}_1)$ acts naturally on $S_{n,a}$ without fixed points. $\tilde{S}_{n,a}$ is the quotient of $S_{n,a}$ by this action and the natural map $S_{n,a} \to \tilde{S}_{n,a}$ makes $S_{n,a}$ a principal locally isotrivial bundle over $\tilde{S}_{n,a}$ with structural group PGL(2) (see Serre [**6**] for this notion).

In the introduction to [**2**] Eisenbud and Van de Ven raised the question of the rationality of $\tilde{S}_{n,a}$. Here we prove the following

THEOREM. *If $a > 0$ and $n - a$ is even, then $\tilde{S}_{n,a}$ is rational.*

The proof of this theorem uses only the construction in [**2**, §5], elementary properties of conic bundles (or $\mathbf{P}_1$-bundles) with smooth fibers and smooth base, and the definition of stably rational varieties due to Kollar and Schreyer [**4**]. An irreducible variety $V$ is said to be stably rational of level $k$ if $V \times \mathbf{P}_k$ is rational. For the elementary properties of conic bundles we need to see Serre [**6**]; we also found useful [**3, 5**].

We write $\tilde{S}_n$ for the variety of smooth, rational curves of degree $n$ in $\mathbf{P}_3$ and $S_n$ for the set of embeddings of degree $n$ of $\mathbf{P}_1$ into $\mathbf{P}_3$. $S_n$ is rational and $S_n \to \tilde{S}_n$ is a principal locally isotrivial bundle with structure group PGL(2). Since $S_n$ (resp. $S_{n,a}$) is rational, if the natural map $p\colon S_n \to \tilde{S}_n$ (resp. $S_{n,a} \to \tilde{S}_{n,a}$) has a rational section,

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then $\tilde{S}_n$ (resp. $\tilde{S}_{n,a}$) is stably rational of level 3. The rationality of $S_{n,a}$ was proved in [**2**, p. 97].

LEMMA 1. *Assume n odd. Then for every $x \in \tilde{S}_n$, there exists a rational section of p defined at x.*

PROOF. Since $\tilde{S}_n$ is contained in Hilb $\mathbf{P}_3$, we have a universal curve $C \to \tilde{S}_n$ with an inclusion $i$: $C \to \tilde{S}_n \times \mathbf{P}_3$ over $\tilde{S}_n$. C is a conic bundle with a smooth base. Since $n$ is odd, this conic bundle is locally trivial in the Zariski topology [**2**]. Thus there is a neighborhood $U$ of $x$ and an $U$-isomorphism $h$: $U \times \mathbf{P}_1 \to C$. The map $i \circ h$ gives the section of $p$ defined on $U$. □

We write $R_n$ for the set of maps of degree $n$ of $\mathbf{P}_1$ into $\mathbf{P}_3$. Again PGL(2) acts on $R_n$ and we write $\tilde{R}_n$ for its quotient. Since we are interested only at birational geometry, there is no problem here; we can substitute $R_n$ with $S_n$ if we want. In [**2**] a key point was the map $G$: $S_{n,a} \to R_{n-a-1}$ $(a > 0)$ constructed in the following way. Fix $f \in S_{n,a}$.

$$N_f := f^*\left(N_{f(\mathbf{P}_1)/\mathbf{P}_3}\right) \cong \mathcal{O}_{\mathbf{P}_1}(2n-1-a) \oplus \mathcal{O}_{\mathbf{P}_1}(2n-1+a)$$

is a quotient of $f^*(T\mathbf{P}_3)$. Thus the subline bundle $\mathcal{O}_{\mathbf{P}_1}(2n-1+a)$ defines a rank-2 subbundle $V_f$ of $f^*(T\mathbf{P}_3)$. The map $G(f)$: $\mathbf{P}_1 \to \mathbf{P}_3$ is constructed by taking for $G(f)(t)$ the plane in $\mathbf{P}_3$ which is determined by $V_{f,t} \subset T\mathbf{P}_{3,f(t)}$. Note that the map $G$ descends to a map $\tilde{G}$: $\tilde{S}_{n,a} \to \tilde{R}_{n-a-1}$ such that, for $0 < a \leqslant n-4$ we have the following commutative diagram:

$$\begin{array}{ccc} S_{n,a} & \xrightarrow{G} & R_{n-a-1} \\ \downarrow q & & \downarrow g \\ \tilde{S}_{n,a} & \xrightarrow{\tilde{G}} & \tilde{R}_{n-a-1} \end{array} \qquad 0 < a \leqslant n-4.$$

Eisenbud and Van de Ven [**2**, p. 97] proved that $G$ is birationally the projection of a product with fiber rational of dimension $2a+5$. If $n-a$ is even, by Lemma 1 $g$ has a rational section. Thus $\tilde{R}_{n-a-1}$ is stably rational of level 3, $\tilde{G}$ is birationally a product with fiber $\mathbf{P}_{2a+5}$ and $\tilde{S}_{n,a}$ is rational. This concludes the proof of Theorem 1.

If $n-a$ is odd, $a > 0$, we do not know very much. A trick easily gives the following

PROPOSITION 1. *Assume $a > 0$. Then $\tilde{S}_{n,a}$ is covered by rational subvarieties of codimension* 2.

PROOF. Fix a point $O \in \mathbf{P}_1$ and a point $P$ in $\mathbf{P}_3$. Let $A_n$ be the set of embeddings $f$ of $\mathbf{P}_1$ into $\mathbf{P}_3$ with $f(O) = P$ and $\deg(f(\mathbf{P}_1)) = n$. $A_n$ is rational. The affine group of projective transformations of $\mathbf{P}_1$ fixing $O$ acts on $A_n$ and let $\tilde{A}_n \subset$ Hilb $\mathbf{P}_3$ be the quotient. $A_n$ is the subset of $S_n$ formed by curves through $P$. The map $A_n \to \tilde{A}_n$ has always a rational section. This follows from the speciality of the affine group [**3**, Lemme 2.3]. Alternatively the restriction to $\tilde{A}_n$ of the conic bundle of Lemma 1 comes from a vector bundle since the point $P$ defines a line bundle on $p^{-1}(\tilde{A}_n)$ with degree one on every fiber.

Thus $\tilde{A}_{n-a-1}$ is stably rational of level 2 and $\tilde{G}|\ \tilde{G}^{-1}(\tilde{A}_{n-a-1})$ has a rational section. Thus $\tilde{G}^{-1}(\tilde{A}_{n-a-1})$ is a rational subvariety of codimension 2 of $\tilde{S}_{n,a}$. □

For $a = 0$ the same method gives only that $\tilde{S}_n$ is covered by codimension 2 subvarieties which are stably rational of level 2.

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