ON THE RATIONALITY OF THE VARIETY OF SMOOTH RATIONAL SPACE CURVES WITH FIXED DEGREE AND NORMAL BUNDLE

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Abstract. Let $\mathcal{S}_{n,a}$ be the variety of smooth, rational curves of degree $n$ in $\mathbb{P}_3$ whose normal bundle has a factor of degree $2n - 1 + a$ and a factor of degree $2n - 1 - a$. In this paper we prove that $\mathcal{S}_{n,a}$ is rational if $n - a$ is even and $a > 0$.

We work over $\mathbb{C}$. Let $\mathcal{S}_{n,a} \subset \text{Hilb} \mathbb{P}_3$ be the set of smooth, rational curves in $\mathbb{P}_3$ of degree $n$ whose normal bundle splits with a summand of degree $2n - 1 - a$ and another of degree $2n - 1 + a$. Eisenbud and Van de Ven [1, 2] proved that for $0 \leq a \leq n - 4$, $\mathcal{S}_{n,a}$ is not empty, irreducible and of dimension $4n - 2a + 1$ (if $a > 0$). Let $\mathcal{S}_{n,a}$ be the set of embeddings $f: \mathbb{P}_1 \to \mathbb{P}_3$ with $f(\mathbb{P}_1) \in \mathcal{S}_{n,a}$. They proved in [2] that $\mathcal{S}_{n,a}$ is irreducible, rational and, if $a > 0$, of dimension $4n - 2a + 4$. $\text{PGL}(2) = \text{Aut}(\mathbb{P}_1)$ acts naturally on $\mathcal{S}_{n,a}$ without fixed points. $\mathcal{S}_{n,a}$ is the quotient of $\tilde{\mathcal{S}}_{n,a}$ by this action and the natural map $\mathcal{S}_{n,a} \to \tilde{\mathcal{S}}_{n,a}$ makes $\mathcal{S}_{n,a}$ a principal locally isotrivial bundle over $\tilde{\mathcal{S}}_{n,a}$ with structural group $\text{PGL}(2)$ (see Serre [6] for this notion).

In the introduction to [2] Eisenbud and Van de Ven raised the question of the rationality of $\tilde{\mathcal{S}}_{n,a}$. Here we prove the following

Theorem. If $a > 0$ and $n - a$ is even, then $\tilde{\mathcal{S}}_{n,a}$ is rational.

The proof of this theorem uses only the construction in [2, §5], elementary properties of conic bundles (or $\mathbb{P}_1$-bundles) with smooth fibers and smooth base, and the definition of stably rational varieties due to Kollar and Schreyer [4]. An irreducible variety $V$ is said to be stably rational of level $k$ if $V \times \mathbb{P}_k$ is rational. For the elementary properties of conic bundles we need to see Serre [6]; we also found useful [3, 5].

We write $\tilde{\mathcal{S}}_n$ for the variety of smooth, rational curves of degree $n$ in $\mathbb{P}_3$ and $\mathcal{S}_n$ for the set of embeddings of degree $n$ of $\mathbb{P}_1$ into $\mathbb{P}_3$. $\mathcal{S}_n$ is rational and $\mathcal{S}_n \to \tilde{\mathcal{S}}_n$ is a principal locally isotrivial bundle with structure group $\text{PGL}(2)$. Since $\mathcal{S}_n$ (resp. $\mathcal{S}_{n,a}$) is rational, if the natural map $p: \mathcal{S}_n \to \tilde{\mathcal{S}}_n$ (resp. $\mathcal{S}_{n,a} \to \tilde{\mathcal{S}}_{n,a}$) has a rational section,
then \( \hat{S}_n \) (resp. \( \hat{S}_{n,a} \)) is stably rational of level 3. The rationality of \( S_{n,a} \) was proved in [2, p. 97].

**Lemma 1.** Assume \( n \) odd. Then for every \( x \in \hat{S}_n \), there exists a rational section of \( p \) defined at \( x \).

**Proof.** Since \( \hat{S}_n \) is contained in \( \text{Hilb } \mathbb{P}_3 \), we have a universal curve \( C \to \hat{S}_n \) with an inclusion \( i: C \to \hat{S}_n \times \mathbb{P}_3 \) over \( \hat{S}_n \). C is a conic bundle with a smooth base. Since \( n \) is odd, this conic bundle is locally trivial in the Zariski topology [2]. Thus there is a neighborhood \( U \) of \( x \) and an \( U \)-isomorphism \( h: U \times \mathbb{P}_1 \to C \). The map \( i \circ h \) gives the section of \( p \) defined on \( U \). \( \square \)

We write \( R_{n} \) for the set of maps of degree \( n \) of \( \mathbb{P}_1 \) into \( \mathbb{P}_3 \). Again \( \text{PGL}(2) \) acts on \( R_{n} \) and we write \( \hat{R}_n \) for its quotient. Since we are interested only at birational geometry, there is no problem here; we can substitute \( R_{n} \) with \( \hat{S}_n \) if we want. In [2] a key point was the map \( G: S_{n,a} \to \mathbb{P}_2^{a} \) constructed in the following way. Fix \( f \in S_{n,a} \).

\[
N_f := f^*(\mathcal{O}_{\mathbb{P}_3}(2n - 1 + a) \otimes \mathcal{O}_{\mathbb{P}_3}(2n - 1 - a))
\]

is a quotient of \( f^*(\mathcal{O}_{\mathbb{P}_3}) \). Thus the subline bundle \( \mathcal{O}_{\mathbb{P}_1}(2n - 1 + a) \) defines a rank-2 subbundle \( V_f \) of \( f^*(\mathcal{O}_{\mathbb{P}_3}) \). The map \( G(f): \mathbb{P}_1 \to \mathbb{P}_3 \) is constructed by taking for \( G(f)(t) \) the plane in \( \mathbb{P}_3 \) which is determined by \( V_f \subset \mathcal{O}_{\mathbb{P}_3}(f(t)) \). Note that the map \( G \) descends to a map \( \hat{G}: \hat{S}_{n,a} \to \hat{R}_{n-a-1} \) such that, for \( 0 < a \leq n - 4 \) we have the following commutative diagram:

\[
\begin{array}{ccc}
S_{n,a} & \xrightarrow{G} & R_{n-a-1} \\
\downarrow q & & \downarrow g \\
\hat{S}_{n,a} & \xrightarrow{\hat{G}} & \hat{R}_{n-a-1}
\end{array}
\]

Eisenbud and Van de Ven [2, p. 97] proved that \( G \) is birationally the projection of a product with fiber rational of dimension \( 2a + 5 \). If \( n - a \) is even, by Lemma 1 \( g \) has a rational section. Thus \( \hat{R}_{n-a-1} \) is stably rational of level 3, \( \hat{G} \) is birationally a product with fiber \( \mathbb{P}_2^{a+5} \) and \( \hat{S}_{n,a} \) is rational. This concludes the proof of Theorem 1.

If \( n - a \) is odd, \( a > 0 \), we do not know very much. A trick easily gives the following

**Proposition 1.** Assume \( a > 0 \). Then \( \hat{S}_{n,a} \) is covered by rational subvarieties of codimension 2.

**Proof.** Fix a point \( O \in \mathbb{P}_1 \) and a point \( P \) in \( \mathbb{P}_3 \). Let \( A_n \) be the set of embeddings \( f \) of \( \mathbb{P}_1 \) into \( \mathbb{P}_3 \) with \( f(O) = P \) and deg(\( f(\mathbb{P}_1) \)) = \( n \). \( A_n \) is rational. The affine group of projective transformations of \( \mathbb{P}_1 \) fixing \( O \) acts on \( A_n \) and let \( \hat{A}_n \subset \text{Hilb } \mathbb{P}_3 \) be the quotient. \( \hat{A}_n \) is the subset of \( A_n \) formed by curves through \( P \). The map \( A_n \to \hat{A}_n \) has always a rational section. This follows from the speciality of the affine group [3, Lemme 2.3]. Alternatively the restriction to \( \hat{A}_n \) of the conic bundle of Lemma 1 comes from a vector bundle since the point \( P \) defines a line bundle on \( p^{-1}(\hat{A}_n) \) with degree one on every fiber.
Thus $\tilde{A}_{n-a-1}$ is stably rational of level 2 and $\tilde{G} | \tilde{G}^{-1}(\tilde{A}_{n-a-1})$ has a rational section. Thus $\tilde{G}^{-1}(\tilde{A}_{n-a-1})$ is a rational subvariety of codimension 2 of $\tilde{S}_{n,a}$. 

For $a = 0$ the same method gives only that $\tilde{S}_n$ is covered by codimension 2 subvarieties which are stably rational of level 2.

REFERENCES


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