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CENTRAL ZERO DIVISORS IN GROUP ALGEBRAS

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ABSTRACT. A central element of the complex group algebra $\mathbf{C}G$ which is a zero divisor in the W^* group algebra $W(G)$ is also a zero divisor in $\mathbf{C}G$. As a corollary, if K is a field of characteristic zero, G is a group, A is an abelian normal subgroup of G, and R is the Ore localization of KG obtained by inverting all nonzero elements of KA, then all matrix rings over R are directly finite and R has the invariant basis property.

The reader is referred to [**1**] and the references therein for terminology not explained here.

THEOREM. *If z is a central element of $\mathbf{C}G$ which is a zero divisor in $W(G)$, then z is a zero divisor in $\mathbf{C}G$.*

Since a zero divisor in a group algebra must, in fact, be a zero divisor in the group algebra of its support group, and similarly for the corresponding W^* algebras (cf. the reductions in the proof of the theorem), we obtain

COROLLARY 1. *If H is a subgroup of G and z is a central element of $\mathbf{C}H$ which is a zero divisor in $W(G)$, then z is a zero divisor in $\mathbf{C}H$.*

Since the group algebra of a free abelian group has no nonzero zero divisors, we also have

COROLLARY 2. *If A is a nontrivial, torsion-free, abelian subgroup of G, then every nonzero element of $\mathbf{C}A$ is a nonzero divisor in $W(G)$.*

The methods of [**1**] then apply to give

COROLLARY 3. *Let K be a commutative integral domain of characteristic zero, G a group, A an abelian normal subgroup of G, and $R = (KG)[Y^{-1}]$, where $Y = KA - \{0\}$. Then all matrix rings $M_n(R)$ are directly finite, and R satisfies the invariant basis property.*

PROOF OF THEOREM. Elements of $W(G)$ may be expressed in the form $a = \sum_{g\in G} a_g g$. Multiplication is convolution: if $b = \sum_{g\in G} b_g g$, then

$$ab = \sum_{\sigma\in G}\left(\sum_{g\in G} a_g b_{g^{-1}\sigma}\right)\sigma.$$

Such a sum a is in $W(G)$ if and only if $ab \in L^2(G)$ for all $b \in L^2(G)$, where $b \in L^2(G)$ is identified with $\sum b(g)g$ and ab is defined as above. The involution on $W(G)$ is given by $a^* = \sum \bar{a}_g g^{-1}$.

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Let $a = \sum a_g g \in W(G)$ be such that $za = 0$. Without loss of generality, $a_1 \neq 0$. (1 will denote the identity of every group appearing.) Let H be the subgroup of G generated by the support of z. It then follows from the above remarks and grouping by cosets that $za' = 0$ where $0 \neq a' = \sum_{g \in H} a_g g \in W(H)$.

Since z is central, H is an FC group: every element has only finitely many conjugates. Thus it suffices to consider the case when G is a finitely generated FC group. Such groups have the property that $G^+ = \{g \in G | g \text{ has finite order}\}$ is a finite normal subgroup such that G/G^+ is torsion-free abelian [**2**, Lemma 4.1.5(iii)].

Let $\overline{g}_1, \ldots, \overline{g}_n$ be a fixed set of free abelian generators for G/G^+. Let $T = \{\alpha \in \mathbf{C} \mid |\alpha| = 1\}$, the circle group. Each $\gamma = (\gamma_1, \ldots, \gamma_n) \in T^n$ defines a homomorphism γ: $G/G^+ \to \mathbf{C}^*$ by $\gamma(\overline{g}_1^{\nu_1} \cdots \overline{g}_n^{\nu_n}) = \gamma_1^{\nu_1} \cdots \gamma_n^{\nu_n}$. This lifts to a homomorphism $G \to \mathbf{C}^*$ which we also denote by γ. This in turn defines an automorphism of $W(G)$ sending $a = \sum a_g g$ to $a^\gamma = \sum a_g \gamma(g) g$. (This is a group action of T^n on $W(G)$.)

Since z is a zero divisor in $W(G)$, $ze = 0$ for some nonzero projection $e = e^* = e^2 \in W(G)$. Since z is central, e is central.

LEMMA 1. *There is an* $\varepsilon > 0$ *such that* $ee^\gamma \neq 0$ *whenever* $\gamma \in T^n$ *satisfies* $|\gamma - 1| < \varepsilon$. ($|\ |$ *denotes the usual norm in* $\mathbf{C}^n$: $|\gamma| = (\sum |\gamma_i|^2)^{1/2}$.)

PROOF. Let $e = \sum e_g g$. Then the coefficient of the identity in ee^γ is

$$(ee^\gamma)_1 = \sum_g e_{g^{-1}} \gamma(g) e_g = \sum \gamma(g) |e_g|^2.$$

It is enough to produce ε so that this is nonzero whenever $|\gamma - 1| < \varepsilon$. Now

$$e_1 = \sum_g |e_g|^2 > 0.$$

Since this converges, there is a finite subset S of G such that

$$\sum_{g \notin S} |e_g|^2 < \frac{e_1}{3}.$$

Since each $|\gamma(g)| = 1$,

$$\left| \sum_{g \notin S} \gamma(g) |e_g|^2 \right| \leq \sum_{g \notin S} |e_g|^2 < \frac{e_1}{3}.$$

Now

$$|e_1 - (ee^\gamma)_1| \leq \sum_{g \in S} |1 - \gamma(g)| \, |e_g|^2 + \sum_{g \notin S} |e_g|^2 + \sum_{g \notin S} \gamma(g) |e_g|^2,$$

so it is enough to find ε so that

$$|1 - \gamma(g)| < \frac{e_1}{3|S| \, |e_g|^2}$$

for all $g \in S$. If the image of g in G/G^+ is $\overline{g}_1^{\nu_1} \cdots \overline{g}_n^{\nu_1}$, then $\gamma(g) = \gamma_1^{\nu_1} \cdots \gamma_n^{\nu_n}$, which is a continuous function of γ. Hence, since S is finite, the desired ε exists.

LEMMA 2. *Let* $\overline{x}_1, \dots, \overline{x}_m$ *be distinct elements of* G/G^+. *Given* $\varepsilon > 0$, *there exist* $\gamma^{(1)}, \dots, \gamma^{(m)} \in T^n$ *with* $|1-\gamma^{(i)}| < \varepsilon$, $i = 1, 2, \dots, m$, *such that* $\det(\gamma^{(i)}(\overline{x}_j)) \neq 0$.

PROOF. The induction on m as in [**2**, Lemma 4.3.3] will apply if we can show that given $\varepsilon > 0$ and $\alpha_1, \dots, \alpha_m \in \mathbf{C}$, not all zero, there exists γ with $|1-\gamma| < \varepsilon$ such that $\alpha_1\gamma(\overline{x}_1) + \cdots + \alpha_m\gamma(\overline{x}_m) \neq 0$. This just amounts to showing that a nonzero polynomial in $\gamma_1, \dots, \gamma_n$ takes on a nonzero value in every neighborhood of 1 in T^n. This follows easily by a similar induction on n.

Continuing the proof of the Theorem, let $z = z_1x_1 + \cdots + z_mx_m$ where $z_i \in \mathbf{C}G^+$ and the x_i are in distinct cosets of G modulo G^+. Applying Lemma 1 inductively (recalling that e, hence every e^γ, is central), there exists $\varepsilon > 0$ such that $e^{\gamma^{(1)}}e^{\gamma^{(2)}}\cdots e^{\gamma^{(m)}} \neq 0$ whenever $|1-\gamma^{(i)}| < \varepsilon$, $i = 1, 2, \dots, m$. Therefore by Lemma 2, there exist $\gamma^{(1)}, \dots, \gamma^{(m)} \in T$ such that $e_0 = e^{\gamma^{(1)}}e^{\gamma^{(2)}}\cdots e^{\gamma^{(m)}} \neq 0$ and $\det(\gamma^{(i)}(\overline{x}_j)) \neq 0$, where $\overline{x}_j$ is the coset of $x_j \bmod G^+$. For any γ, z^γ is central and $z^\gamma e^\gamma = 0$. Therefore $e_0z^{\gamma^{(i)}} = z^{\gamma^{(i)}}e_0 = 0$, $i = 1, 2, \dots, m$. Since $z^{\gamma^{(i)}} = \gamma^{(i)}(\overline{x}_1)z_1x_1 + \cdots + \gamma^{(i)}(\overline{x}_m)z_mx_m$ and $\det(\gamma^{(i)}(\overline{x}_j)) \neq 0$, it follows that $e_0z_ix_i = 0$, hence $e_0z_i = 0$, $i = 1, 2, \dots, m$. As before, this implies there is a nonzero $c \in W(A)$, where A is the subgroup generated by the supports of $z_1, z_2, \dots, z_m$, with $cz_i = 0$, $i = 1, 2, \dots, m$. But $A \subset G^+$ is finite, so $W(A) = \mathbf{C}A \subseteq \mathbf{C}G$. Thus $0 \neq c \in \mathbf{C}G$ and $cz = 0$.

REMARKS. 1. If G is finitely generated free abelian, the theorem may be proved via Fourier transforms. Briefly, a suitable version of the Plancherel Theorem says that (i) the idempotent e acting on $L^2(G)$ corresponds to the operator in $L^2(\hat{G}) = L^2(T^n)$ given by multiplication by the characteristic function χ_A of some set A, and (ii) $z = \sum \varsigma_g g$ corresponds to the function $\hat{z}(\gamma) = \sum \varsigma_i\gamma(g)$. The equation $ze = 0$ translates to $\hat{z}\chi_A = 0$, which implies $\hat{z}(\gamma) = 0$ for all $\gamma \in A$. Since z is a finite sum, A must have measure zero, forcing $e = 0$. The more algebraic proof given above was motivated by this Fourier transform argument as follows. If A has positive measure, its intersection with a translate by γ sufficiently close to 1 must intersect A in a set of positive measure. The characteristic function of the translate corresponds to e^γ; the characteristic function of the intersection corresponds to ee^γ.

2. The referee has pointed out that it is, in fact, possible to reduce to the case of a finitely-generated, torsion-free, abelian group as follows. The finitely-generated FC group has a finitely-generated, torsion-free, central subgroup N of finite index. Letting $Y = KN - \{0\}$, $R = (HK)[Y^{-1}]$ is artinian. Thus if z is not a zero divisor in R, z has an inverse xz_0^{-1} in R, where $z_0 \in Y$. Since z is a zero divisor in $W(G)$, z_0 is a zero divisor in $W(G)$, hence in $W(N)$. This permits a slight simplification in the notation of the remainder of the proof.

3. We have learned since submitting this paper for publication that S. Rosset has independently given a proof (using trigonometric polynomials) of Corollary 3.

REFERENCES

1. K. R. Goodearl, *Localized group rings, the invariant basis property and Euler characteristics*, Proc. Amer. Math. Soc. **91** (1984), 523–528.

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