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RATIOS OF DUAL GENERIC DEGREES OF A FINITE COXETER GROUP

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ABSTRACT. A formula is obtained for the ratio of dual generic degrees of a finite Coxeter group.

1. Introduction. The purpose of this paper is to determine the ratio of the generic degrees corresponding to dual irreducible characters of a finite Coxeter group. These ratios could, of course, be found on a case by case basis, since the generic degrees are now known (see [**2**, **1**]). We obtain a uniform formula for all cases (Corollary 2 below), which expresses the ratio in terms of the character table of the Coxeter group.

The present investigation started while the author was working with G. Lusztig on the generic degrees for type H_4 [**1**]. See §4 for a general result related to such calculations. The proof of Theorem 1 below was inspired by the proof of Lemma 1.11 of Lusztig's paper [**9**], which establishes the one variable case. The present methods apply to generic degrees in several variables.

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2. A preliminary result. Throughout this paper (W,S) is a finite Coxeter system, with set of distinguished generators S and length function $l\colon W \to \mathbf{N}$. Let R be the set of W-conjugates of elements of S, and let $\{u_C\}$ be a set of indeterminates, one for each conjugacy class $C \subseteq R$. Define $u_s = u_C$ if $s \in S \cap C$. Let K be a field containing $\mathbf{Q}(\{u_C\})$. The *generic ring* A of (W,S) over K is the K-algebra with basis elements T_w, $w \in W$, and multiplication determined by

$$
\begin{aligned}
&T_s^2 = u_sT_1 + (u_s-1)T_s && \text{if } s \in S,\\
&T_wT_{w'} = T_{ww'} && \text{if } l(w)+l(w') = l(ww').
\end{aligned}
\tag{1}
$$

It is known that A is a separable K-algebra with unit element T_1.

Let $s(1)\cdots s(l)$, $l = l(w)$, be a reduced expression for $w \in W$. Define $\varepsilon_w = \operatorname{sign}(w) = (-1)^l$, $u_w = u_{s(1)}\cdots u_{s(l)}$, and $\hat{T}_w = \varepsilon_w u_w T^{-1}_{w^{-1}}$. Then

$$
\hat{T}_w = \varepsilon_w(T_{s(1)} - u_{s(1)} + 1)\cdots(T_{s(l)} - u_{s(l)} + 1) \tag{2}
$$

by (1). The K-linear extension $a \to \hat{a}$ of $T_w \to \hat{T}_w$ is an involutory automorphism of A (see [**6**]).

If $a,b \in A$ and $ab = \sum_w \alpha_w T_w$, $\alpha_w \in K$, set $B(a,b) = \alpha_1$. It is known that B is a symmetric, associative nondegenerate K-bilinear form on A, with dual bases

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$\{T_w\}$, $\{u_w^{-1}T_{w^{-1}}\}$ [**4**, 5.12]. Following [**7**] we write

$$T_{y^{-1}}^{-1} = \sum_{x\in W} u_x^{-1}\overline{R_{x,y}}T_x$$

for $\overline{R_{x,y}} \in K$. Note $\overline{R_{x,y}} = B(T_{x^{-1}}, T_{y^{-1}}^{-1})$.

Let w_0 be the longest element of W. If $x \in W$, then $l(x) + l(x^{-1}w_0) = l(w_0)$. Therefore

$$\varepsilon_x\varepsilon_{x^{-1}w_0} = \varepsilon_{w_0}, \quad u_xu_{x^{-1}w_0} = u_{w_0}, \quad \text{and} \quad T_xT_{x^{-1}w_0} = T_{w_0}.$$

We require the following result, which is similar to [**7**, 2.1(iv)].

LEMMA 1. $\overline{R_{w_0y,w_0x}} = \overline{R_{x^{-1},y^{-1}}}$.

PROOF. We have $\overline{R_{w_0y,w_0x}} = B(T_{y^{-1}w_0}, T_{x^{-1}w_0}^{-1})$. Since

$$T_{y^{-1}w_0} = T_y^{-1}T_{w_0} \quad \text{and} \quad T_{x^{-1}w_0}^{-1} = T_{w_0}^{-1}T_x,$$

$$B(T_{y^{-1}w_0}, T_{x^{-1}w_0}^{-1}) = B(T_y^{-1}, T_x).$$

As $B(T_y^{-1}, T_x) = \overline{R_{x^{-1},y^{-1}}}$, the proof is complete.

3. The main results. In this section we assume that K is a splitting field for A. Let V be a simple A-module, with representation ρ: $A \to \mathrm{End}_K(V)$ and character $\chi \in \mathrm{Irr}\, A$. The *generic degree* d_χ is defined by

$$d_\chi = \frac{\dim V \cdot P}{\sum_{w\in W} u_w^{-1}\chi(T_w)\chi(T_{w^{-1}})},$$

where $P = \sum_w u_w$ is the Poincaré polyomial of (W, S) [**3**]. The primitive central idempotent of A corresponding to χ is

$$e_\chi = \frac{d_\chi}{P}\sum_{w\in W} u_w^{-1}\chi(T_{w^{-1}})T_w \tag{3}$$

[**5**, 9.17]. Define an endomorphism $E(\rho)$ of V by

$$E(\rho) = \sum_{w\in W} u_w^{-1}\chi(T_{w^{-1}})\rho(T_{w^{-1}w_0}^{-1}). \tag{4}$$

The *dual* $\hat\rho$ of ρ is given by $\hat\rho(a) = \rho(\hat a)$, $a \in A$. Note that $\hat\rho$ is an irreducible representation of A with character $\hat\chi(a) = \chi(\hat a)$. If χ corresponds to $\chi_0 \in \mathrm{Irr}\, W$ under a specialization $u_s \to 1$ (see [**4**, 7.1]), then $\hat\chi$ corresponds to sign $\otimes\, \chi_0$ under the same specialization by (2). Let $E(\hat\rho)$ be defined as in (4), with $\hat\rho, \hat\chi$ replacing ρ, χ.

LEMMA 2. $E(\hat\rho) = \varepsilon_{w_0}E(\rho)$.

PROOF. Since $(T_{y^{-1}w_0}^{-1})\widehat{\ } = \varepsilon_{w_0y}u_{w_0y}^{-1}T_{w_0y}$, we have

$$\begin{aligned} E(\hat\rho) &= \sum_{y\in W} u_y^{-1}\chi(\varepsilon_yu_yT_y^{-1})\rho(\varepsilon_{w_0y}u_{w_0y}^{-1}T_{w_0y}) \\ &= \varepsilon_{w_0}\sum_{x,y\in W} u_x^{-1}\overline{R_{x^{-1},y^{-1}}}\chi(T_{x^{-1}})\rho(u_{w_0y}^{-1}T_{w_0y}). \end{aligned}$$

The proof is completed by applying Lemma 1 and the identity

$$\sum_{y\in W} u_{w_0y}^{-1}\overline{R_{w_0y,w_0x}}T_{w_0y} = T_{x^{-1}w_0}^{-1}.$$

THEOREM 1. *Suppose V is a simple A-module with character χ. Then $T_{w_0}^2$ acts on V as the scalar $u_{w_0}(d_{\hat{\chi}}/d_\chi)$.*

PROOF. Since $T_{w^{-1}w_0}^{-1} = T_{w_0}^{-1}T_w$ and $\rho(ae_\chi) = \rho(a)$, $a \in A$, we have $E(\rho) = (P/d_\chi)\rho(T_{w_0}^{-1})$ by (3) and (4). Similarly, $E(\hat{\rho}) = (P/d_{\hat{\chi}})\hat{\rho}(T_{w_0}^{-1})$. Also, $\hat{\rho}(T_{w_0}^{-1}) = \varepsilon_{w_0}u_{w_0}^{-1}\rho(T_{w_0})$. The theorem now follows from Lemma 2.

COROLLARY 1 (KILMOYER [**8**, PROPOSITION 14(ii)]). *$T_{w_0}^2$ is central in A.*

For C a conjugacy class in R, define

$$e(\chi, C) = |C|\chi_0(s)/\dim V, \qquad s \in S\cap C,$$

where $\chi_0(s) = \chi(T_s)_{u_s=1}$. Then $e(\chi, C)$ is a rational integer [**5**, 9.31]. Let $w_0 = s_1s_2\cdots s_N$, $s_i \in S$, be a reduced expression for w_0. Then $T_{w_0} = T_{s_1}\cdots T_{s_N}$. It is known that the number of s_i $(1 \le i \le N)$ in a given class $C \subseteq R$ is exactly $|C|$. Arguing as in [**3**], we see that the multiplicity of u_s as an eigenvalue of $\rho(T_s)$ is $(\chi_0(s) + \chi_0(1))/2$. Therefore

$$\det\rho(T_{w_0}^2) = \left(u_{w_0}\prod_C u_C^{e(\chi,C)}\right)^{\dim V}.$$

Thus $T_{w_0}^2$ acts on V as a scalar of the form $\lambda u_{w_0}\prod_C u_C^{e(\chi,C)}$, with λ a root of unity. This scalar must assume the value 1 when each indeterminate u_C is replaced by 1 since $w_0^2 = 1$, and therefore $\lambda = 1$. Hence we have $\rho(T_{w_0}^2) = u_{w_0}\prod_C u_C^{e(\chi,C)}$, a formula first observed by Springer (see [**3**]). This, together with the theorem, proves the following.

COROLLARY 2. *$d_{\hat{\chi}}/d_\chi = \prod_C u_C^{e(\chi,C)}$, the product taken over all conjugacy classes $C \subseteq R$.*

4. An application. In this section we derive a recursion formula giving certain of the generic degrees for W in terms of those for the standard parabolic subgroups $W_J = \langle J\rangle$, $J \subseteq S$, $J \ne S$. A key ingredient in the formula is the ratio $d_{\hat{\chi}}/d_\chi$, which may be found using the character table for W (Corollary 2).

Let A_J be the subalgebra of A spanned by the T_w for $w \in W_J$. We assume K is a splitting field for all A_J, $J \subseteq S$. If $\chi \in \operatorname{Irr} A$ and $\theta \in \operatorname{Irr} A_J$, let $m(\chi, \theta)$ be the multiplicity of θ in the restriction $\operatorname{Res}_{A_J}^A(\chi)$ of χ to A_J (or, equivalently, of θ_0 in $\operatorname{Res}_{W_J}^W(\chi_0)$). We have

$$\frac{P}{P_J}d_\theta = \sum_{\chi\in\operatorname{Irr} A} m(\chi,\theta)d_\chi, \tag{5}$$

where P_J is the Poincaré polynomial for (W_J, J). This is established by Surowski in [**10**] for the crystallographic cases. In general, (5) may be easily proved using the fact that the generic degrees d_χ are the unique solution to the following linear system (see [**1**]).

$$\sum_{\chi} \chi(T_w)d_\chi = \begin{cases} P & \text{if } w = 1, \\ 0 & \text{if } w \neq 1. \end{cases}$$

We also have

$$\sum_{\substack{J \subseteq S \\ \theta \in \operatorname{Irr} A_J}} (-1)^{|J|} m(\chi, \theta) m(\chi', \theta) = \delta_{\hat{\chi}, \chi'} \tag{6}$$

for χ, $\chi' \in \operatorname{Irr} A$. This corresponds, on the level of W, to the fact that

$$\sum_{J \subseteq S} (-1)^{|J|} \operatorname{Ind}_{W_J}^W \operatorname{Res}_{W_J}^W(\chi_0) = \operatorname{sign} \bigotimes \chi_0$$

for $\chi_0 \in \operatorname{Irr} W$. Combining (5) and (6), we obtain

$$d_{\hat{\chi}} = \sum_{\substack{J \subseteq S \\ \theta \in \operatorname{Irr} A_J}} (-1)^{|J|} \frac{P}{P_J} m(\chi, \theta) d_\theta.$$

Hence

$$\left(\frac{d_{\hat{\chi}}}{d_\chi} - (-1)^{|S|} \right) d_\chi = \sum_{\substack{J \neq S \\ \theta \in \operatorname{Irr} A_J}} (-1)^{|J|} \frac{P}{P_J} m(\chi, \theta) d_\theta. \tag{7}$$

If $|S|$ is odd or $d_{\hat{\chi}} \neq d_\chi$, we may divide by $(d_{\hat{\chi}}/d_\chi) - (-1)^{|S|}$ to solve (7) for d_χ. This is the desired (partial) recursion formula.

In practice, a slightly different technique has been used. Using (5) and transitivity of induction, the right-hand side of (7) may be replaced by a sum

$$\sum_{\substack{I \\ \theta \in \operatorname{Irr} A_I}} (-1)^{|I|} \frac{P}{P_I} n(\chi, \theta) d_\theta,$$

where I ranges over the maximal subsets of S. The coefficients $n(\chi, \theta)$ are certain integers, and are not unique. They may be found by solving the underdetermined linear system

$$\hat{\chi} - (-1)^{|S|}\chi = \sum_{\substack{I \\ \theta \in \operatorname{Irr} A_K}} (-1)^{|I|} n(\chi, \theta) \operatorname{Ind}_{A_I}^A(\theta)$$

(see, for example, [**2**]).

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