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$C(K, E)$ CONTAINS A COMPLEMENTED COPY OF c_0

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ABSTRACT. Let E be a Banach space and let K be a compact Hausdorff space. We denote by $C(K, E)$ the Banach space of all E-valued continuous functions defined on K, endowed with the supremum norm. We prove in this paper that if K is infinite and E is infinite-dimensional, then $C(K, E)$ contains a complemented subspace isomorphic to c_0.

Let E be a Banach space and let K be a compact Hausdorff space. We denote by $C(K, E)$ ($C(K)$ if E is the scalar field) the Banach space of all E-valued continuous functions defined on K with the supremum norm. The notations and terminology used here can be found in [**3**, **5**, **7**].

E. and P. Saab [**8**] showed that if l_1 is isomorphic to a complemented subspace of $C(K, E)$, then l_1 is isomorphic to a complemented subspace of E. They noted that the moral behind their result is: since l_1 is not isomorphic to a complemented subspace of any $C(K)$, if $C(K, E)$ contains a complemented copy of l_1, then E must contain a complemented copy of l_1. They also noted that similar remarks can be made about other previous results.

We will prove in this paper that (unless trivial situations exist) $C(K, E)$ always contains a complemented subspace isomorphic to c_0. So our result differs from the one above, because we can find Banach spaces E and spaces $C(K)$ that contain no complemented copy of c_0 and such that $C(K, E)$ contains a complemented copy of c_0.

THEOREM. *Let K be an infinite compact Hausdorff space and E an infinite-dimensional Banach space. Then $C(K, E)$ contains a complemented subspace isomorphic to c_0.*

PROOF. Since E is infinite-dimensional there exists a sequence (x_n^*) in E^*, with $\|x_n^*\| = 1$ for all n, such that (x_n^*) is w^*-convergent to zero [**6**]. On the other hand, since K is infinite there exists a sequence (G_n) of open nonempty pairwise disjoint subsets of K. Let (x_n) be a sequence in E such that

$$\langle x_n, x_n^* \rangle = 1 \quad \text{and} \quad \|x_n\| \leq 2 \quad \text{for } n \in \mathbf{N}.$$

For each $n \in \mathbf{N}$ choose $t_n \in G_n$. We define $T: C(K, E) \to l_\infty$ by $T(\phi) = (\langle \phi(t_n), x_n^* \rangle)$, it is clear that T is a bounded linear operator. Notice that if $f \in C(K)$ and $x \in E$ then $T(f(\cdot)x) = (f(t_n)x_n^*(x))$, but for each n, $|f(t_n)x_n^*(x)| \leq \|f\| \, |x_n^*(x)|$. Therefore $\lim f(t_n)x_n^*(x) = 0$. This shows that T is c_0 valued because the linear span of the set $\{f(\cdot)x: f \in C(K),\ x \in E\}$ is dense in $C(K, E)$. For each $n \in \mathbf{N}$ take $f_n \in C(K)$ such that

$$f_n(t_n) = 1, \quad f_n(K \backslash G_n) = \{0\} \quad \text{and} \quad \|f_n\| = 1.$$

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If σ is a finite subset of $\mathbf{N}$, then we have

$$\left\|\sum_{n\in\sigma} f_n(\cdot)x_n\right\| = \sup_{t\in K}\left\|\sum_{n\in\sigma} f_n(t)x_n\right\| = \sup_{t\in\bigcup_{n\in\sigma}G_n} \|f_n(t)x_n\| \leq 2.$$

Therefore $\sum_{n=1}^{\infty} f_n(\cdot)x_n$ is a weakly unconditionally convergent series in $C(K,E)$, but $\sum_{n=1}^{\infty} T(f_n(\cdot)x_n)$ is not an unconditionally convergent series in c_0 because $T(f_n(\cdot)x_n) = e_n$ for all $n \in \mathbf{N}$, where (e_n) denotes the unit vector basis of c_0. Consequently T is not an unconditionally converging operator.

By [**1**] there exists a subspace H of $C(K,E)$ isomorphic to c_0 such that $T|_H$ (the restriction of T to H) is an isomorphism onto $T(H)$. The subspaces of c_0 isomorphic to c_0 are complemented in c_0 [**9**], so $T(H)$ is complemented in c_0. If $P_1: c_0 \to T(H)$ is a continuous projection onto $T(H)$, then the map $P = (T|_H)^{-1} \circ P_1 \circ T$ is a continuous projection from $C(K,E)$ onto H. This proves that $C(K,E)$ contains a complemented subspace isomorphic to c_0.

DEFINITION. A Banach space F is called a Grothendieck space if every bounded linear operator $T: F \to c_0$ is weakly compact.

As a consequence of our theorem we have the following result of Khurana [**4**].

COROLLARY 1 (KHURANA). *If $C(K,E)$ is a Grothendieck space, then either K is finite or E is finite dimensional.*

Another consequence is the following

COROLLARY 2. *A space $C(K)$ is a Grothendieck space if and only if $C(K)$ contains no complemented copy of c_0.*

PROOF. Suppose that $C(K)$ contains no complemented copy of c_0 and let $T: C(K) \to c_0$ be a bounded linear operator. If T is not weakly compact, then T is not unconditionally converging [**7**, Theorem 1] and by the proof of our theorem, $C(K)$ would contain a complemented copy of c_0. This contradiction finishes the proof.

EXAMPLE. There exist Banach spaces E and spaces $C(K)$ that contain no complemented copies of c_0 and such that $C(K,E)$ contains a complemented copy of c_0.

Let $K = \beta\mathbf{N}$ be the Stone-Čech compactification of the integers. Then $C(K) \simeq l_\infty$ is a Grothendieck space and does not contain a complemented copy of c_0, while $C(K, C(K)) \simeq C(K \times K)$ contains a complemented copy of c_0 by our theorem.

This example is not as surprising as it looks since W. Rudin (see [**2**]) showed that the product of two compact infinite spaces is never Stonean. In fact, Rudin's result can be deduced from the main theorem of this paper. Namely the following is true.

COROLLARY 3. *Let K_1 and K_2 be two compact Hausdorff spaces. If $C(K_1 \times K_2)$ does not contain a complemented copy of c_0 then K_1 or K_2 must be finite.*

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