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ON THE HANS LEWY EXTENSION PHENOMENON IN HIGHER CODIMENSION

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ABSTRACT. In this work the authors extend the results of their paper entitled *Families of analytic discs in* $\mathbf{C}^n$ *with boundaries on a prescribed CR submanifold*. It is proven that a nongeneric CR manifold M whose Lewy form is not identically zero can be extended to a manifold $\tilde{M}$ of one higher dimension, which is foliated by analytic discs. Moreover this result is used to prove that a sufficiently smooth CR function f on M extends to a function $\tilde{f}$ which is CR on $\tilde{M}$.

1. Introduction. This work was inspired by the work of H. Lewy (see [**5–7**]). He proved the following theorem: let S be a sufficiently smooth real hypersurface in $\mathbf{C}^N$ ($N \geqslant 2$) whose Levi form at the origin does not vanish identically, then there is an open set Ω in $\mathbf{C}^N$, lying on one side of S, with $S \cap \bar{\Omega}$ a neighborhood of the origin in S, such that any sufficiently smooth function f on $S \cap \bar{\Omega}$, which satisfies the tangential Cauchy-Riemann equations to S there, has a unique extension to a holomorphic function $\tilde{f}$ in Ω with $\tilde{f}|_{S\cap\bar{\Omega}} = f$. In this work we prove that if the hypersurface S is replaced by a real, CR submanifold M in $\mathbf{C}^N$ whose codimension is greater than one, then there exists a CR "manifold-with-boundary" $\tilde{M}$ with $\tilde{M} \cap M$ a neighborhood of the origin in M, such that any sufficiently smooth function f on $\tilde{M} \cap M$, which satisfies the tangential Cauchy-Riemann equations to M there, has a unique extension to a function $\tilde{f}$ which satisfies the tangential Cauchy-Riemann equations to $\tilde{M}$ (see Theorem 2).

2. Preliminaries. We shall use the same notation as in [**3**]. Our manifolds M will be embedded CR manifolds, i.e. $M \subset \mathbf{C}^N$ and if $p \in M$ the complex dimension of the largest complex subspace of $T_p(M)$, $T_p(M)$ being the (real) tangent space to M at p, is a constant independent of p. This constant is called the CR dimension of M and the CR codimension of M is defined to be the dimension of M minus twice its CR dimension. A manifold $M \subset \mathbf{C}^N$ will be called of type (m, l) if the CR dimension of $M = m$ and the CR codimension of $M = l$. Thus a manifold of type (m, l) has dimension $d = 2m + l$. Let $k \equiv N - (l + m)$. We will call a CR manifold *generic* if and only if $k = 0$, i.e. if and only if the codimension of $M =$ the CR codimension of M. We deal with the nongeneric case.

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We can locally express any embedded CR manifold M of dimension d contained in $\mathbf{C}^N$ by a system of $q = 2N - d$ (real) equations, $\rho_i(z) = 0$, $1 \leq i \leq q$, where $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$ and $d\rho_1 \wedge d\rho_2 \wedge \cdots \wedge d\rho_q \neq 0$ on M. The space of holomorphic tangent vectors at p in M is characterized by

$$HT_p(M) = \left\{ X \equiv \sum_{j=1}^{N} a_j \frac{\partial}{\partial z_j} \,\Big|\, \sum_{j=1}^{N} a_j \frac{\partial \rho_i}{\partial z_j}(p) = 0,\ 1 \leq i \leq q \right\},$$

where the a_j are complex numbers. The antiholomorphic tangent space $\overline{HT_p(M)}$, is defined by complex conjugation. A differentiable function f on M is a *CR function on M* if and only if $Xf(p) = 0$ for every $X \in \overline{HT_p(M)}$ and every $p \in M$.

By the Levi form of M at p we will mean the following vector-valued form L_p: $HT_p(M) \to N_p(M)$, where $N_p(M)$ is the space of normal vectors to M at p. Choosing the functions $\rho_1, \ldots, \rho_q$ above so that at $p \in M$, $d\rho_1(p), \ldots, d\rho_q(p)$ are orthonormal and letting $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbf{C}^N$ denote the components of the holomorphic tangent vector $Z(p)$ we can explicitly define the Levi form of M at p by

$$\text{(2.1)} \qquad \begin{cases} L_p(Z) = \displaystyle\sum_{i=1}^{q} \left[\sum_{j,k=1}^{N} 4 \frac{\partial^2 \rho_i}{\partial z_j \partial \bar{z}_k}(p) \zeta_j \zeta_k \right] d\rho_i(p), \\ \displaystyle\sum_{j=1}^{N} \frac{\partial \rho_i}{\partial z_j}(p) \zeta_j = 0, \qquad 1 \leq i \leq q. \end{cases}$$

A differentiable transformation between two CR manifolds, ψ: $X \to Y$, $X \subset C^r$, $Y \subset C^s$ is said to be a *CR map at $p \in X$* if the complexified differential satisfies $d\psi_p$: $HT_{p'}(X) \to HT_{\psi(p')}(Y)$ for all p' in a neighborhood of p in X. We say that ψ is a *CR map on X* if it is a CR map at each point $p \in X$. Moreover, if ψ has components, $\psi_1, \ldots, \psi_s$ then ψ is a CR map if and only if each component ψ_j is a CR function on X (see [**4**]). We have the following proposition (see [**4**] for a proof).

PROPOSITION 1. *Let M_1 be a CR manifold $M_1 \subset \mathbf{C}^{N_1}$ of type (m, l) such that M_1 can be written as a graph over its tangent space at some point in M_1. Let ψ: $M_1 \to \mathbf{C}^s$ and let $M_2 \subset C^{N_1+S}$ be the graph of ψ over M_1. Then M_2 is a CR manifold of the same type (m, l) if and only if the function $\phi = (I, \psi)$: $M_1 \to M_2$ is a CR map; i.e. if and only if each component of ψ is a CR function of M_1.*

We shall also rely heavily on the following slight modification of a theorem of Baoendi and Treves [**1**] (see [**4**] for the modified proof).

PROPOSITION 2. *Let $M \subset \mathbf{C}^N$ be an embedded CR manifold of type (m, l) and class C^2, $2 \leq s \leq \infty$. Then there exists an open neighborhood U of the origin in $\mathbf{C}^N$ such that for any f which is a CR function on M of class C^t, $2 \leq t \leq s \leq \infty$, there exists a sequence of polynomials p^j such that $p^j \to f$ in $C^{t-1}(U \cap M)$.*

We shall also use the fact that any nongeneric CR manifold M' of type (m, l) contained in $\mathbf{C}^N = \mathbf{C}^l \times \mathbf{C}^m \times \mathbf{C}^k$ can be expressed (locally) as the image by a CR map of a generic manifold $M \subset \mathbf{C}^n = \mathbf{C}^l \times \mathbf{C}^n$. We recall the argument from [**4**] and fix notation for use in §§3 and 4. For, without loss of generality, we can assume that

$0 \in M'$, $T_0(M') = \mathbf{R}^l \times \mathbf{C}^m \times 0 \cong \mathbf{R}^l \times \mathbf{C}^m$ and, for U' a sufficiently small neighborhood of the origin in $\mathbf{C}^N$, we have

$$(2.2)\quad M' = \begin{cases} (z_1,\dots,z_N) \in U' \,|\, y_\eta = h_\eta(x_1,\dots,x_l, z_{l+1},\dots,z_n), & \eta = 1,\dots,l, \\ z_{n+\nu} = [h'_\nu + ih''_\nu](x_1,\dots,x_l, z_{l+1},\dots,z_n), & \nu = 1,\dots,1, \end{cases}$$

where h_η, h'_ν, h''_ν are real-valued functions defined in some neighborhood of the origin Ω' in $\mathbf{R}^l \times \mathbf{C}^m$. Let π: $\mathbf{C}^N \times \mathbf{C}^k \to \mathbf{C}^n$ be a projection onto the first n components and define $M = \pi(M')$. Then, restricting U' and Ω' if necessary, we have that M is a generic CR manifold of type (m, l) in $\mathbf{C}^n$ defined by

(2.3)

$$M = \{(z_1,\dots,z_n) \in \pi(U') \subset \mathbf{C}^n \,|\, y_\eta = h_\eta(x_1,\dots,x_l, z_{l+1},\dots,z_n), n = 1,\dots,l\}.$$

The function $\phi \equiv \{\pi|_{M'}\}^{-1}$: $M \to M' \subset \mathbf{C}^N$ can be expressed by

$$\phi_\eta(z_1,\dots,z_n) = \begin{cases} z_\eta, & \eta \leqslant n, \\ [h'_{\eta-n} + ih''_{\eta-n}](x_1,\dots,x_l, z_{l+1},\dots,z_n), & n+1 \leqslant \eta \leqslant N, \end{cases}$$

for $(z_1,\dots,z_n) \in M$ and $\eta = 1,\dots,N$. We have by Proposition 1 that ϕ is a CR map and thus each ϕ_η is a CR function on M.

3. On going up one dimension in the nongeneric case. Our main theorem, stated below, extends the results of §9 [**3**, Theorem 9.1, p. 364] to the nongeneric case.

THEOREM 1. *Let M be a real d-dimensional CR manifold of type (m, l) embedded in $\mathbf{C}^N$. Let $\xi \neq 0$ be a Levi vector at some point $p \in M$. Then (with the amount of differentiability stated below) there exists a local, real $d+1$-dimensional embedded CR manifold-with-boundary $\tilde{M}$ of type $(m+1, l-1)$ such that the boundary of $\tilde{M}$ is equal to an open neighborhood of p in M, and with $T_p(\tilde{M}) = \operatorname{span}\{T_p(M), \xi\}$. Moreover, $\tilde{M}$ is foliated by a real $d-1$ parameter family of complex one-dimensional analytic discs with their boundaries on M:*

(i) *If M is of class C^β and $\beta \geqslant 5$, then $\tilde{M}$ is of class $C^{[(\beta-2)/3],1/2}$.*

(ii) *If M is real analytic then $\tilde{M}$ is a real analytic; moreover, $\tilde{M}$ has a "border" $\tilde{M}_\delta - \tilde{M}$ in the sense that $\tilde{M}$ extends real analytically to a slightly larger $\tilde{M}_\delta$ (also foliated) such that $M \subset \tilde{M}_\delta$ forms an embedded real analytic hyerpsurface in $\tilde{M}_\delta$.*

(iii) *If M is of class C^∞, then $\tilde{M}$ is of class C^∞.*

PROOF. The case $k = 0$ is exactly Theorem 9.1 proved in [**3**]. Therefore we assume $k > 0$. In the proof it will also be convenient to change notation from M, N, ξ to M', N, ξ'. As in [**3**], we can assume that we have chosen a convenient coordinate system so that

$$\begin{cases} p = 0, \\ T_0(M') = \mathbf{R}^l \times \mathbf{C}^m \times \{0\} = \{(x, w, 0)\}, \\ N_0(M') = i\mathbf{R} \times \{0\} \times \mathbf{C}^k = \{(iy, 0, z')\}, \\ HT_0(M') = \{0\} \times \mathbf{C}^m \times \{0\} = \{(0, w, 0)\}, \\ \xi' = dy_1 = L'_0(\partial/\partial w_1), \end{cases}$$

where $x = (x_1, \ldots, x_l)$, $y = (y_1, \ldots, y_l)$, $z = x + iy$, $w = (w_1, \ldots, w_m)$, $z' = (z'_1, \ldots, z'_k)$. Defining $M = \pi(M')$ as above in (2.3) we have that $T_0(M) = T_0(M')$, $HT_0(M) = HT_0(M')$, $N_0(M) = i\mathbf{R}^l \times \{0\} = \{(iy, 0)\}$ and using 2.1 we have that

$$\xi \equiv L_0\left(\frac{\partial}{\partial w_1}\right) = L'_0\left(\frac{\partial}{\partial w_1}\right) = \xi'.$$

As stated in §2, we have that ϕ: $M \to M'$ is a CR diffeomorphism and thus each component of ϕ_ν is a CR function on M. By (2.3) we have that ϕ_ν, $1 \leqslant \nu \leqslant n$, are each of the identity functions on M. The modified result of Bavendi and Treves, Proposition 2, yields that there exists U, a neighborhood of the origin in $\mathbf{C}^n$, and sequences of holomorphic polynomials $\{f_\nu^j\}$, $1 \leqslant \nu \leqslant k$, such that for each ν, $f_\nu^j \to \phi_{n+\nu}$ on $C^{\beta-1}(U \cap M)$. Since $\phi_{n+\nu}(0) = d\phi_{n+\nu}(0) = 0$ and $\beta > 5$ we can assume that the f_ν^j were chosen such that they all vanish to second order at the origin. Since M is generic we can, using Theorem 9 of [**3**], construct $\tilde{M}$, an $(m+1, l-1)$ CR manifold as in Theorem 1. Thus $\tilde{M}$ is of class $C^{[(\beta-2)/3],1/2}$ and is foliated by analytic discs with boundaries on M. By the maximum principle we have that for each ν, $\{f_\nu^j\}$ is uniformly Cauchy on each leaf of the foliation of $\tilde{M}$. Restricting our attention to a slightly smaller compact submanifold of M (which contains 0) we have that if we define $\tilde{\phi}_{n+\nu}$ to be the $\lim_{j\to\infty} f_\nu^j$ on $\tilde{M}$, then $\tilde{\phi}_{n+\nu} \in C^0(\tilde{M})$ and $\tilde{\phi}_{n+\nu}|_M = \phi_{n+\nu}$. Moreover, since M is generic any vector field of the form D_{z_η}, $1 \leqslant \eta \leqslant n$, is in the complex linear space of vector fields tangential to M (see [**4**, Lemma 3.1]). Therefore, for each fixed ν, $1 \leqslant \nu \leqslant k$, $\{D_{z_\eta} f_\eta^j\}$ is uniformly Cauchy in $C^{\beta-2}(M)$. Continuing by induction, we have that $\{D_z^\alpha f_\nu^j\}$ is uniformly Cauchy on $C(M)$ for all $|\alpha| \leqslant \beta - 1$, ν fixed. Applying the maximum principle we have that $\{D_z^\alpha f_\nu^j\}$ is uniformly Cauchy on $C(\tilde{M})$ for $1 \leqslant \eta \leqslant n$, ν fixed, and $|\alpha| \leqslant \beta - 1$. Since $\tilde{M}$ is at least of class $C^{[(\beta-2)/3],1/2}$, we have $f_\nu^j \to \phi_{n+\nu}$ on $C^{[(\beta-2)3,1/2}(\tilde{M})$. In particular, if $X \in \overline{HT}_p(\tilde{M})$, since $(\beta-2)/3 \geqslant 1$, we have $Xf_\nu^j \to X\tilde{\phi}_{n+\nu}$ on $\tilde{M}$ and since $Xf_\nu^j \equiv 0$ on $\tilde{M}$ we have that $\tilde{\phi}_{n+\nu}$ is CR on $\tilde{M}$, for each $1 \leqslant \nu \leqslant k$.

Let $\tilde{\phi} = (I_n, \tilde{\phi}_{n+1}, \ldots, \tilde{\phi}_N)$, then $\tilde{\phi}|_M = \phi$ and defining $\tilde{M}' \equiv \tilde{\phi}(\tilde{M})$ we have that $\tilde{M}'$ is the graph of $(\tilde{\phi}_{n+1}, \ldots, \tilde{\phi}_N)$ over M. By Proposition 1 we have that $\tilde{M}'$ is a CR manifold of type $(m+1, l-1)$. Moreover, since $\tilde{M}$ is foliated by a real $d-1$ parameter family of complex one-dimensional analytic submanifolds (discs) with their boundaries on M and $\tilde{\phi}$ is CR on $\tilde{M}$, we have that $\tilde{M}'$ is also foliated by a real $d-1$ parameter family of discs. Since the boundary of $\tilde{M} \subset M$, we have that the boundary of $\tilde{M}'$ = boundary of $\tilde{\phi}(\tilde{M}) \subset \tilde{\phi}(M) = \phi(M) = M'$. All we need prove now to complete cases (i) and (iii) is that

$$T_0(\tilde{M}') = T_0(\tilde{M}) = \operatorname{span}(T_0(M), \xi) = \operatorname{span}(T_0(M'), \xi')$$

which is equivalent to proving $d\tilde{\phi}_{n+\nu}(0) = 0$. This follows since $f_\nu^j \to \tilde{\phi}_{n+\nu}$ in $C^1(\tilde{M})$ and each f_ν^j vanish to second order at the origin.

For the real analytic case we have, using Tomassini [**8**] or Lemma 2.3 of [**2**], that $\phi_{n+\nu}$ is the trace of M of a holomorphic function. Applying Theorem 8.2 of [**3**] to $\phi_{n+\nu}$ yields the extension $\tilde{\phi}_{n+\nu}$ which is holomorphic in a neighborhood of $\tilde{M}_\delta$. Defining $\tilde{\phi} = (I_n, \tilde{\phi}_{n+1}, \ldots, \tilde{\phi}_N)$, $\tilde{M}' = \tilde{\phi}(\tilde{M})$ and $\tilde{M}'_\delta = \tilde{\phi}(\tilde{M}_\delta)$ yields the above result.

4. The extension of CR functions. We now state and prove Theorem 2. It is, in fact, a corollary of Theorem 1 but it is so interesting in its own right, that we feel it deserves the status of theorem.

THEOREM 2. *Let $M \subset \mathbf{C}^N$ be a CR manifold of class C^β, $\beta \geqslant 5$, and let $\xi \neq 0$ be a Levi vector at some point $p \in M$. Let f be a C^k, $2 \leqslant k \leqslant \beta$, function defined and satisfying the tangential Cauchy-Riemann equations to M in a neighborhood of p. Then there exists a neighborhood U of $p \in M$ such that $f|_{U \cap M}$ can be extended to a CR function $\tilde{f}$ of class C^μ, where $\mu = \min(k-1, \{[(\beta-2)/3], 1/2\})$, defined on the CR manifold $\tilde{M}$ whose existence is proved in Theorem 1. This extension is unique and if f and M are both C^∞ or real analytic, then $\tilde{f}$ is also of the same class.*

PROOF. We can consider M to be of type (m, l) and to be in the form (2.2). Moreover, we can consider $\tilde{M}$ to be constructed as in Theorem 1. Thus the space of tangent vectors to M and $\tilde{M}$ is in the complex linear space of $dz_1, \ldots, dz_n$, where $n = m + l$. From Proposition 2 we have that there exists $\tilde{U} \subset \mathbf{C}^N$ and polynomials p_j defined on $\tilde{U}$ such that $p_j \to f$ in $C^{k-1}(M \cap \tilde{U})$. As above, the maximum principle yields a unique function $\tilde{f}$ such that $p_j \to \tilde{f}$ in $C(\tilde{M} \cap \tilde{U})$. Moreover, given D_{z_η}, $1 \leqslant \eta \leqslant n$, $D^\alpha_{z_\eta} p_j \to f$ in $C(M \cap \tilde{U})$ for $|\alpha| \leqslant k-1$ and thus by the maximum principle $\{D^\alpha_{z_\eta} p_j\}$ is a Cauchy sequence in $\tilde{M}$ But this means that $p_j \to \tilde{f}$ on $C^\mu(\tilde{M} \cap \tilde{U})$ with μ as above. Since $\mu \geqslant 1$ and the polynomials are holomorphic, we have that $\tilde{f}$ is CR on $\tilde{M} \cap \tilde{U}$.

The final remark about the smoothness clearly follows from Theorem 1 cases (ii) and (iii).

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