ON THE HANS LEWY EXTENSION PHENOMENON
IN HIGHER CODIMENSION

C. D. HILL¹ AND G. TAIANI²

ABSTRACT. In this work the authors extend the results of their paper entitled Families of analytic discs in Cⁿ with boundaries on a prescribed CR submanifold. It is proven that a nongeneric CR manifold M whose Lewy form is not identically zero can be extended to a manifold ¯¯M of one higher dimension, which is foliated by analytic discs. Moreover this result is used to prove that a sufficiently smooth CR function f on M extends to a function ¯f which is CR on ¯M.

1. Introduction. This work was inspired by the work of H. Lewy (see [5–7]). He proved the following theorem: let S be a sufficiently smooth real hypersurface in Cᴺ (N ≥ 2) whose Levi form at the origin does not vanish identically, then there is an open set Ω in Cᴺ, lying on one side of S, with S ∩ ¯Ω a neighborhood of the origin in S, such that any sufficiently smooth function f on S ∩ Ω, which satisfies the tangential Cauchy-Riemann equations to S there, has a unique extension to a holomorphic function ¯f in Ω with ¯f|S∩Ω = f. In this work we prove that if the hypersurface S is replaced by a real, CR submanifold M in Cᴺ whose codimension is greater than one, then there exists a CR “manifold-with-boundary” ¯M with ¯M ∩ M a neighborhood of the origin in M, such that any sufficiently smooth function f on ¯M ∩ M, which satisfies the tangential Cauchy-Riemann equations to M there, has a unique extension to a function ¯f which satisfies the tangential Cauchy-Riemann equations to ¯M (see Theorem 2).

2. Preliminaries. We shall use the same notation as in [3]. Our manifolds M will be embedded CR manifolds, i.e. M ⊂ Cᴺ and if p ∈ M the complex dimension of the largest complex subspace of Tᵖ(M), Tᵖ(Cᴺ) being the (real) tangent space to M at p, is a constant independent of p. This constant is called the CR dimension of M and the CR codimension of M is defined to be the dimension of M minus twice its CR dimension. A manifold M ⊂ Cᴺ will be called of type (m, l) if the CR dimension of M = m and the CR codimension of M = l. Thus a manifold of type (m, l) has dimension d = 2m + l. Let k = N − (l + m). We will call a CR manifold generic if and only if k = 0, i.e. if and only if the codimension of M = the CR codimension of M. We deal with the nongeneric case.

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We can locally express any embedded CR manifold $M$ of dimension $d$ contained in $\mathbb{C}^N$ by a system of $q = 2N - d$ (real) equations, $\rho_i(z) = 0$, $1 \leq i \leq q$, where $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ and $d\rho_1 \wedge d\rho_2 \wedge \cdots \wedge d\rho_q \neq 0$ on $M$. The space of holomorphic tangent vectors at $p$ in $M$ is characterized by

$$HT_p(M) = \left\{ X = \sum_{j=1}^N a_j \frac{\partial}{\partial z_j} \left| \sum_{j=1}^N a_j \frac{\partial \rho_j}{\partial z_j}(p) = 0, 1 \leq i \leq q \right\},$$

where the $a_j$ are complex numbers. The antiholomorphic tangent space $HT^*_p(M)$ is defined by complex conjugation. A differentiable function $f$ on $M$ is a CR function on $M$ if and only if $Xf(p) = 0$ for every $X \in HT^*_p(M)$ and every $p \in M$.

By the Levi form of $M$ at $p$ we will mean the following vector-valued form $L_p: HT_p(M) \to N_p(M)$, where $N_p(M)$ is the space of normal vectors to $M$ at $p$. Choosing the functions $\rho_1, \ldots, \rho_q$ above so that at $p \in M$, $d\rho_1(p), \ldots, d\rho_q(p)$ are orthonormal and letting $(\xi_1, \xi_2, \ldots, \xi_q) \in \mathbb{C}^N$ denote the components of the holomorphic tangent vector $Z(p)$ we can explicitly define the Levi form of $M$ at $p$ by

$$L_p(Z) = \sum_{i=1}^q \left| \sum_{j,k=1}^N 4 \frac{\partial^2 \rho_j}{\partial z_j \partial \overline{z}_k} (p) \xi_j \overline{\xi}_k \right| d\rho_i(p),$$

(2.1)

A differentiable transformation between two CR manifolds, $\psi: X \to Y$, $X \subset \mathbb{C}'$, $Y \subset \mathbb{C}^s$ is said to be a CR map at $p \in X$ if the complexified differential satisfies $d\psi_p: HT_p(X) \to HT_{\psi(p)}(Y)$ for all $p'$ in a neighborhood of $p$ in $X$. We say that $\psi$ is a CR map on $X$ if it is a CR map at each point $p \in X$. Moreover, if $\psi$ has components, $\psi_1, \ldots, \psi_q$, then $\psi$ is a CR map if and only if each component $\psi_j$ is a CR function on $X$ (see [4]). We have the following proposition (see [4] for a proof).

**Proposition 1.** Let $M_1$ be a CR manifold $M_1 \subset \mathbb{C}^N$ of type $(m, l)$ such that $M_1$ can be written as a graph over its tangent space at some point in $M_1$. Let $\psi: M_1 \to \mathbb{C}'$ and let $M_2 \subset \mathbb{C}^{N+5}$ be the graph of $\psi$ over $M_1$. Then $M_2$ is a CR manifold of the same type $(m, l)$ if and only if the function $\phi = (I, \psi): M_1 \to M_2$ is a CR map; i.e. if and only if each component of $\psi$ is a CR function of $M_1$.

We shall also rely heavily on the following slight modification of a theorem of Baouendi and Treves [1] (see [4] for the modified proof).

**Proposition 2.** Let $M \subset \mathbb{C}^N$ be an embedded CR manifold of type $(m, l)$ and class $C^2$, $2 \leq s \leq \infty$. Then there exists an open neighborhood $U$ of the origin in $\mathbb{C}^N$ such that for any $f$ which is a CR function on $M$ of class $C'$, $2 \leq t \leq s \leq \infty$, there exists a sequence of polynomials $p^j$ such that $p^j \to f$ in $C^{s-1}(U \cap M)$.

We shall also use the fact that any nongeneric CR manifold $M'$ of type $(m, l)$ contained in $\mathbb{C}^N = \mathbb{C}^l \times \mathbb{C}^m \times \mathbb{C}^k$ can be expressed (locally) as the image by a CR map of a generic manifold $M \subset \mathbb{C}^n = \mathbb{C}^l \times \mathbb{C}^n$. We recall the argument from [4] and fix notation for use in §§3 and 4. For, without loss of generality, we can assume that
$0 \in M', \, T_0(M') = \mathbb{R}^r \times \mathbb{C}^m \times \{0\} \cong \mathbb{R}^r \times \mathbb{C}^m$ and, for $U'$ a sufficiently small neighborhood of the origin in $\mathbb{C}^N$, we have

$$M' = \left\{ (z_1, \ldots, z_N) \in U' \left| \begin{array}{l}
\eta = 1, \ldots, l,
\eta = 1, \ldots, l,
\end{array} \right. \right\}$$

where $h_\eta$, $h'_\eta$, $h''_\eta$ are real-valued functions defined in some neighborhood of the origin $\tilde{\Omega}'$ in $\mathbb{R}^r \times \mathbb{C}^m$. Let $\pi: \mathbb{C}^N \rightarrow \mathbb{C}^n$ be a projection onto the first $n$ components and define $M = \pi(M')$. Then, restricting $U'$ and $\Omega'$ if necessary, we have that $M$ is a generic CR manifold of type $(m, l)$ in $\mathbb{C}^n$ defined by

$$M = \left\{ (z_1, \ldots, z_n) \in \pi(U') \subset \mathbb{C}^n \left| \begin{array}{l}
y = h_\eta(x_1, \ldots, x_l, z_{l+1}, \ldots, z_n),\quad \eta = 1, \ldots, l,
y = h_\eta(x_1, \ldots, x_l, z_{l+1}, \ldots, z_n),\quad \eta = 1, \ldots, l,
\end{array} \right. \right\}.$$ 

The function $\phi = \left\{ \pi \mid_{M'} \right\}^{-1}: M \rightarrow M' \subset \mathbb{C}^N$ can be expressed by

$$\phi_\eta(z_1, \ldots, z_n) = \left\{ \begin{array}{l}
\eta 
\eta
\end{array} \right\}$$

for $(z_1, \ldots, z_n) \in M$ and $\eta = 1, \ldots, N$. We have by Proposition 1 that $\phi$ is a CR map and thus each $\phi_\eta$ is a CR function on $M$.

3. On going up one dimension in the nongeneric case. Our main theorem, stated below, extends the results of §9 [3, Theorem 9.1, p. 364] to the nongeneric case.

**Theorem 1.** Let $M$ be a real $d$-dimensional CR manifold of type $(m, l)$ embedded in $\mathbb{C}^N$. Let $\xi \neq 0$ be a Levi vector at some point $p \in M$. Then (with the amount of differentiability stated below) there exists a local, real $d + 1$-dimensional embedded CR manifold-with-boundary $\tilde{M}$ of type $(m + 1, l - 1)$ such that the boundary of $\tilde{M}$ is equal to an open neighborhood of $p$ in $M$, and with $T_p(\tilde{M}) = \text{span}(T_p(M), \xi)$. Moreover, $\tilde{M}$ is foliated by a real $d - 1$ parameter family of complex one-dimensional analytic discs with their boundaries on $M$:

(i) If $M$ is of class $C^B$ and $B \geq 5$, then $\tilde{M}$ is of class $C^{((B - 2)/3), 1/2}$. 

(ii) If $M$ is real analytic then $\tilde{M}$ is a real analytic; moreover, $\tilde{M}$ has a "border" $\tilde{M}_8 - \tilde{M}$ in the sense that $\tilde{M}$ extends real analytically to a slightly larger $\tilde{M}_8$ (also foliated) such that $M \subset \tilde{M}_8$ forms an embedded real analytic hypersurface in $\tilde{M}_8$. 

(iii) If $M$ is of class $C^\infty$, then $\tilde{M}$ is of class $C^\infty$. 

**Proof.** The case $k = 0$ is exactly Theorem 9.1 proved in [3]. Therefore we assume $k > 0$. In the proof it will also be convenient to change notation from $M, N, \xi$ to $M', N, \xi'$. As in [3], we can assume that we have chosen a convenient coordinate system so that

$$\begin{align*}
p &= 0, \\
T_0(M') &= \mathbb{R}^r \times \mathbb{C}^m \times \{0\} = \{(x, w, 0)\}, \\
N_0(M') &= i\mathbb{R} \times \{0\} \times \mathbb{C}^k = \{(iy, 0, z')\}, \\
HT_0(M') &= \{0\} \times \mathbb{C}^m \times \{0\} = \{(0, w, 0)\}, \\
\xi' &= dy_1 = L_0(\partial/\partial w_1),
\end{align*}$$

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where \( x = (x_1, \ldots, x_t) \), \( y = (y_1, \ldots, y_t) \), \( z = x + iy \), \( w = (w_1, \ldots, w_m) \), \( z' = (z'_1, \ldots, z'_k) \). Defining \( M = \pi(M') \) as above in (2.3) we have that \( T_0(M) = T_0(M') \), \( HT_0(M) = HT_0(M') \), \( N_0(M) = i\mathbb{R} \times \{0\} = \{(iy, 0)\} \) and using 2.1 we have that

\[
\xi \equiv L_0 \left( \frac{\partial}{\partial w_i} \right) = L_0' \left( \frac{\partial}{\partial w_i} \right) = \xi'.
\]

As stated in §2, we have that \( \phi: M \to M' \) is a CR diffeomorphism and thus each component of \( \phi \) is a CR function on \( M \). By (2.3) we have that \( \phi_n, 1 \leq n \leq n', \) are each of the identity functions on \( M \). The modified result of Bavendi and Trèves, Proposition 2, yields that there exists \( U \), a neighborhood of the origin in \( \mathbb{C}^n \), and sequences of holomorphic polynomials \( \{f_j\} \), \( 1 \leq n \leq k \), such that for each \( n \), \( f_j^n \to \phi_{n, +v} \) on \( \mathbb{C}^{n-1} \cap M \). Since \( \phi_{n, +v} (0) = d\phi_{n, +v} (0) = 0 \) and \( \beta > 5 \) we can assume that the \( f_j^n \) were chosen such that they all vanish to second order at the origin. Since \( M \) is generic we can, using Theorem 9 of [3], construct \( \tilde{M} \), an \( (m + 1, l - 1) \) CR manifold as in Theorem 1. Thus \( \tilde{M} \) is of class \( C^{(\beta - 2)/3, 1/2} \) and is foliated by analytic discs with boundaries on \( M \). By the maximum principle we have that for each \( n \), \( \{f_j^n\} \) is uniformly Cauchy on each leaf of the foliation of \( \tilde{M} \). Restricting our attention to a slightly smaller compact submanifold of \( M \) (which contains \( 0 \) we have that if we define \( \tilde{\phi}_{n, +v} \) to be the limit \( \lim_{j \to \infty} f_j^n \) on \( M' \), then \( \tilde{\phi}_{n, +v} \in C^0(\tilde{M}) \) and \( \tilde{\phi}_{n, +v} | M = \phi_{n, +v} \). Moreover, since \( M \) is generic any vector field of the form \( D_{z^n} \), \( 1 \leq n \leq n' \), is in the complex linear space of vector fields tangential to \( M \) (see [4, Lemma 3.1]). Therefore, for each fixed \( n \), \( 1 \leq n \leq k \), \( \{D_{z^n} f_j^n\} \) is uniformly Cauchy on \( C^{\beta - 2}(M) \). Continuing by induction, we have that \( \{D_{z^n} f_j^n\} \) is uniformly Cauchy on \( C(M) \) for all \( |\alpha| \leq \beta - 1, n \) fixed. Applying the maximum principle we have that \( \{D_{z^n} f_j^n\} \) is uniformly Cauchy on \( C(M) \) for \( 1 \leq n \leq n', \) \( \nu \) fixed, and \( |\alpha| \leq \beta - 1 \). Since \( M \) is at least of class \( C^{(\beta - 2)/3, 1/2} \), we have \( f_j^n \to \phi_{n, +v} \) on \( C^{(\beta - 2)/3, 1/2}(\tilde{M}) \). In particular, if \( X \in HT_p(\tilde{M}) \), since \( (\beta - 2)/3 > 1 \), we have \( Xf_j^n \to X\tilde{\phi}_{n, +v} \) on \( \tilde{M} \) and since \( Xf_j^n \equiv 0 \) on \( \tilde{M} \) we have that \( \tilde{\phi}_{n, +v} \) is CR on \( \tilde{M} \), for each \( 1 \leq n \leq k \).

Let \( \tilde{\phi} = (\tilde{I}, \tilde{\phi}_{n, +1}, \ldots, \tilde{\phi}_{n, n}) \), \( \tilde{\phi} | M = \phi \) and defining \( \tilde{M}' = \tilde{\phi}(M) \) we have that \( \tilde{M}' \) is the graph of \( (\tilde{\phi}_{n, +1}, \ldots, \tilde{\phi}_{n, n}) \) over \( M \). By Proposition 1 we have that \( \tilde{M}' \) is a CR manifold of type \( (n + 1, l - 1) \). Moreover, since \( \tilde{M} \) is foliated by a real \( d - 1 \) parameter family of complex one-dimensional analytic submanifolds (discs) with their boundaries on \( M \) and \( \tilde{\phi} \) is CR on \( \tilde{M} \), we have that \( \tilde{M}' \) is also foliated by a real \( d - 1 \) parameter family of discs. Since the boundary of \( \tilde{M} \subset M \), we have that the boundary of \( \tilde{M}' = \text{boundary of } \tilde{\phi}(M) \subset \tilde{\phi}(M) = \phi(M) = M' \). All we need prove now to complete cases (i) and (iii) is that

\[
T_0(\tilde{M}') = T_0(\tilde{M}) = \text{span}(T_0(M), \xi) = \text{span}(T_0(M'), \xi')
\]

which is equivalent to proving \( d\tilde{\phi}_{n, +v}(0) = 0 \). This follows since \( f_j^n \to \tilde{\phi}_{n, +v} \) in \( C^1(\tilde{M}) \) and each \( f_j^n \) vanish to second order at the origin.

For the real analytic case we have, using Tomassini [8] or Lemma 2.3 of [2], that \( \phi_{n, +v} \) is the trace of \( M \) of a holomorphic function. Applying Theorem 8.2 of [3] to \( \phi_{n, +v} \) yields the extension \( \tilde{\phi}_{n, +v} \) which is holomorphic in a neighborhood of \( \tilde{M}_b \). Defining \( \tilde{\phi} = (\tilde{I}, \tilde{\phi}_{n, +1}, \ldots, \tilde{\phi}_{n, n}) \), \( \tilde{M}' = \tilde{\phi}(\tilde{M}) \) and \( \tilde{M}'_b = \tilde{\phi}(\tilde{M}_b) \) yields the above result.
4. The extension of CR functions. We now state and prove Theorem 2. It is, in fact, a corollary of Theorem 1 but it is so interesting in its own right, that we feel it deserves the status of theorem.

**Theorem 2.** Let $M \subset \mathbb{C}^N$ be a CR manifold of class $C^\beta$, $\beta \geq 5$, and let $\xi \neq 0$ be a Levi vector at some point $p \in M$. Let $f$ be a $C^k$, $2 \leq k \leq \beta$, function defined and satisfying the tangential Cauchy-Riemann equations to $M$ in a neighborhood of $p$. Then there exists a neighborhood $U$ of $p \in M$ such that $f|_U \cap M$ can be extended to a CR function $\tilde{f}$ of class $C^\mu$, where $\mu = \min(k - 1, \{(\beta - 2)/3, 1/2\})$, defined on the CR manifold $\tilde{M}$ whose existence is proved in Theorem 1. This extension is unique and if $f$ and $M$ are both $C^\infty$ or real analytic, then $\tilde{f}$ is also of the same class.

**Proof.** We can consider $M$ to be of type $(m, l)$ and to be in the form (2.2). Moreover, we can consider $\tilde{M}$ to be constructed as in Theorem 1. Thus the space of tangent vectors to $M$ and $\tilde{M}$ is in the complex linear space of $dz_1, \ldots, dz_n$, where $n = m + l$. From Proposition 2 we have that there exists $\hat{U} \subset \mathbb{C}^N$ and polynomials $p_j$ defined on $\hat{U}$ such that $p_j \to f$ in $C^{k-1}(M \cap \hat{U})$. As above, the maximum principle yields a unique function $\tilde{f}$ such that $p_j \to \tilde{f}$ in $C(\tilde{M} \cap \hat{U})$. Moreover, given $D^\alpha$, $1 \leq \eta \leq n$, $D^\alpha p_j \to f$ in $C(M \cap \hat{U})$ for $|\alpha| \leq k - 1$ and thus by the maximum principle $\{D^\alpha p_j\}$ is a Cauchy sequence in $\tilde{M}$ But this means that $p_j \to \tilde{f}$ on $C^\mu(\tilde{M} \cap \hat{U})$ with $\mu$ as above. Since $\mu \geq 1$ and the polynomials are holomorphic, we have that $\tilde{f}$ is CR on $\tilde{M} \cap \hat{U}$.

The final remark about the smoothness clearly follows from Theorem 1 cases (ii) and (iii).

**REFERENCES**


DEPARTMENT OF MATHEMATICS, PACE UNIVERSITY, PACE PLAZA, NEW YORK, NEW YORK 10038