INVERTIBILITY IN NEST ALGEBRAS

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ABSTRACT. Let \( \mathcal{F} \) denote a complete nest of subspaces of a complex Hilbert space \( \mathcal{H} \), and let \( \mathcal{C} \) denote the nest algebra defined by \( \mathcal{F} \). Let \( \mathcal{K} \) denote the ideal of compact operators on \( \mathcal{H} \). If \( \mathcal{F} \) has no infinite-dimensional gaps then \( T \in \mathcal{C} \) is invertible in \( \mathcal{C} \) if and only if it is invertible in \( \mathcal{C} + \mathcal{K} \). An example is given of a nest with an infinite gap for which there exists an operator in \( \mathcal{C} \) which is invertible in \( \mathcal{C} + \mathcal{K} \) but not in \( \mathcal{C} \).

1. Introduction. The idea of looking at a chain of invariant subspaces of an operator and relating it to the structure of the operator and its spectrum is basic to finite-dimensional linear algebra. The first attempt to generalize such an approach for nonselfadjoint operators in Hilbert space seems to have been in a classical paper of M. Lifshitz [6]. This was later pursued by a number of mathematicians in the Soviet Union and is, for the most part, summarized in [5].

Independently, J. Ringrose [7] defined the notion of a nest algebra, an algebra of operators which leaves a chain of subspaces invariant, and studied the basic properties of such algebras. We are concerned with the question of identifying the invertible elements of a nest algebra \( \mathcal{C} \). While it is always of interest to identify the invertible elements of a Banach algebra, in this case the problem has significance in the stability theory of input-output systems. The interested reader is referred to [3, 4].

It was shown in [2] that if \( \mathcal{K} \) represents the ideal of compact operators on \( \mathcal{H} \) and \( \mathcal{C} \) is a nest algebra, then \( \mathcal{C} + \mathcal{K} \) is a Banach algebra.

Here we show, that if a complete nest \( \mathcal{F} \) has only finite-dimensional gaps and \( \mathcal{C} \) is the nest algebra determined by \( \mathcal{F} \), then \( T \in \mathcal{C} \), where \( T^{-1} \in \mathcal{C} + \mathcal{K} \) implies \( T^{-1} \in \mathcal{C} \). This corrects the proof of and extends a previously announced result in this direction [8].

2. Preliminaries. Let \( \mathcal{H} \) be a separable Hilbert space. A family of subspaces \( \mathcal{F} \) of \( \mathcal{H} \) will be called a nest if it is totally ordered by inclusion. \( \mathcal{F} \) is complete if:

(i) \( \{0\}, \mathcal{H} \in \mathcal{F} \);

(ii) given any subnest \( \mathcal{F}_0 \subset \mathcal{F} \), the subspaces \( \bigcap \{L: L \in \mathcal{F}_0 \} \) and \( \bigvee \{L: L \in \mathcal{F}_0 \} \) are both members of \( \mathcal{F} \).

Given a complete nest \( \mathcal{F} \) and a nonzero subspace \( M \) in \( \mathcal{F} \), we define the predecessor \( M_\prec \) of \( M \) by

\[
M_\prec = \bigvee \{L: L \in \mathcal{F}, L \subset M, L \neq M\}.
\]
If $C$ is the family of operators defined by $C = \{A \in \mathcal{B}(\mathcal{H}) : AL \subseteq L, \text{for all } L \in \mathcal{F}\}$, it is easily seen that $C$ is a weakly closed algebra containing the identity. $C$ is called the nest algebra associated with $\mathcal{F}$.

Suppose $K$ is the ideal of compact operators in $\mathcal{B}(\mathcal{H})$. The following is proved in [2].

**Theorem A.** $C + K$ is a norm closed algebra. The natural isomorphism of $C/C \cap K$ and $C + K/C$ is a Banach algebra isomorphism.

3. The main result. Suppose $T \in C$ is an invertible operator on $\mathcal{H}$ whose inverse is in $C + K$. Then $T^{-1} = A + K$ with $A \in C$, $K \in K$. Thus $I = TT^{-1} = TA + TK$, and $TK = I - TA$ is a compact operator in $C$. If $\mathcal{F}$ is a continuous nest ($M_\ast = M$ for all $M \in \mathcal{F}$), it follows from a result of Gohberg-Krein [5] or Ringrose [7] that $TK$ is quasinilpotent. Thus $(I - TK)$ is invertible and $(I - TK)^{-1}$ is in fact a power series in $TK$. It follows that $(I - TK)^{-1}$ and, therefore $T^{-1} = A(I - TK)^{-1}$, is in $C$.

This argument does not work if $\mathcal{F}$ is not continuous, since $I - TK$ may not be invertible. We will show that if for all $M \in \mathcal{F}$, $\dim(M \ominus M_\ast) < \infty$, then the same property holds. We begin with a preliminary lemma.

**Lemma 1.** Suppose $T$ is an operator on $\mathcal{H}$ such that the restriction of $T$ to an invariant subspace of finite codimension is invertible. If $T$ has trivial kernel, then $T$ is invertible on $\mathcal{H}$.

**Proof.** Suppose $\mathcal{H} = M \oplus M^\perp$ and $T$ has the matrix representation

\[
T = \begin{pmatrix}
T_1 & T_2 \\
0 & T_3
\end{pmatrix}
\]

with respect to this decomposition. Then, it suffices to show that $T_3 = (I - P_M)T(I - P_M)$ is invertible on $M^\perp$. Since $M^\perp$ is finite dimensional we show $T_3$ has trivial kernel. If $x \in M^\perp$ such that $T_3x = 0$, this implies $Tx \in M$. But then $Tx = T_1y = Ty$ for some $y \in M$. But this is impossible since $T$ has trivial kernel and $x \in M^\perp$.

**Theorem 2.** Let $\mathcal{F}$ be a complete nest on $\mathcal{H}$ with no infinite-dimensional gaps. If $C$ is the nest algebra determined by $\mathcal{F}$ then $T \in C$ and $T^{-1} \in C + K$ implies $T^{-1} \in C$.

**Proof.** Suppose $M \in \mathcal{F}$. Since, as above, $TA = I - TK$ implies $TK \in C$ and $TK$ is compact, it is enough to consider the case where $P_M(I - TK)|M$ is not invertible. For otherwise, $(I - TK)M = M$ implies $M = TAM$ and therefore $M \supset AM = T^{-1}M$.

If $P_M(I - TK)|M$ is not invertible, then, since $TK$ is compact, there exist at most finitely many \{\{L_i\}_{i=1}^n\} $\in \mathcal{F}$ with $L_i \neq L_i^\ast$ such that $L_i \ominus L_i^\ast$ contains a vector $x_i$ with $(I - TK)x_i \in L_i^\ast$. Assume $L_1 \subset L_2 \subset \cdots \subset L_n \subset M$.

If $(L_1)_\ast = \{0\}$, then $L_1$ is a finite-dimensional subspace invariant under $T$ and thus, by a simple dimension argument, $TL_1 = L_1$.

If $(L_1)_\ast \neq \{0\}$, then by the assumption on $\{L_i\}_{i=1}^n$, $P(L_1)_\ast((I - TK)|(L_1)_\ast$ is invertible. By the argument of the first paragraph of the proof, we obtain $T(L_1)_\ast = (L_1)_\ast$. Applying Lemma 1 to $T$ gives $TL_1 = L_1$. 

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Now consider $L_2$, which we write as $L_2 = (L_2 \Join L_1) \Join L_1$. Let $\hat{S}$ denote the compression of an operator $S \in \mathcal{C}$ to $L_1^\perp$. Since $TL_1 = L_1$, to show $TL_2 = L_2$ it suffices to show that $\hat{T}(L_2 \Join L_1) = L_2 \Join L_1$.

If $L_1 = (L_2)_\perp$, then $L_2 \Join L_1$ is finite dimensional. Noting that $T$ is invertible and $TL_1 = L_1$ implies $\hat{T}$ is invertible, and that $\hat{T}(L_2 \Join L_1) \subset (L_2 \Join L_1)$, it follows that $\hat{T}(L_2 \Join L_1) = L_2 \Join L_1$.

If $L_1 \neq (L_2)_\perp$, then $\hat{T}(L_2 \Join L_1) = [(L_2)_\perp \Join L_1]$ is invertible. Thus by the argument given in the first paragraph of the proof, $\hat{T}(L_2)_\perp \Join L_1 = [(L_2)_\perp \Join L_1]$. By Lemma 1, applied to $\hat{T}$ we obtain $\hat{T}(L_2 \Join L_1) = L_2 \Join L_1$ and thus $TL_2 = L_2$.

Since $M = (M \Join L_n) \Join (L_n \Join L_{n-1}) \Join \cdots \Join (L_2 \Join L_1) \Join L_1$, the above argument applied $n$ times gives $T^M = M$ and completes the proof.

What happens when $\mathcal{F}$ has infinite-dimensional gaps? Before we answer this we give a result to show that this question for compact operators is equivalent to the question for finite rank operators. This may be of independent interest.

**Theorem 3.** Suppose that for $T \in \mathcal{C}$, $T^{-1} = A + F$, $F$ finite rank, implies $F \in \mathcal{C}$. Then $T \in \mathcal{C}$ and $T^{-1} = A + K$, $K$ compact, implies $K \in \mathcal{C}$.

**Proof.** Suppose $T \in \mathcal{C}$ with $T^{-1} = A + K$. We show that for every $\varepsilon > 0$, where $\varepsilon < 1/\|T\|$, there exists an invertible operator $B_\varepsilon^{-1} = L + F$ with $L \in \mathcal{C}$, $F$ finite rank, $B_\varepsilon \in \mathcal{C}$ and $\|B_\varepsilon^{-1} - T^{-1}\| < \varepsilon$. Since, by hypothesis, $B_\varepsilon^{-1} \in \mathcal{C}$, this will imply $T^{-1} \in \mathcal{C}$.

If $\|B_\varepsilon^{-1} - T^{-1}\| < 1/\|T\|$, it follows that $B_\varepsilon = T\sum_{i=0}^{\infty} (I - B_\varepsilon^{-1}T)^i$. Thus if $B_\varepsilon^{-1}T \in \mathcal{C}$ so is $B_\varepsilon$. As we have seen, $T^{-1} = A + K$ implies $KT \in \mathcal{C}$. Thus [1] there exists a sequence $\{F_n\}$ of finite rank operators in $\mathcal{C}$ such that $\|F_n - KT\| \to 0$ as $n \to \infty$.

Let $S_n = A + F_nT^{-1}$. Then $S_n - T^{-1} = F_nT^{-1} - K$ and, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$, $\|S_n - T^{-1}\| < \varepsilon$. If $\varepsilon < 1/(\|T\|)$, it follows that $S_n$ is invertible and

$$S_nT = AT + F_nT^{-1}T = AT + F_n \in \mathcal{C}.$$

Thus given $\varepsilon > 0$, choose $B_\varepsilon^{-1} = S_n$ for $n > N(\varepsilon)$. This completes the proof.

**Example 1.** Let $\mathcal{H} = l^2(0, \infty) \oplus l^2(0, \infty)$ with the nest whose projections are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} U & e_0 \otimes e_0 \\ 0 & U^* \end{pmatrix}$$

where $e_0 \otimes e_0$ is the projection on the one-dimensional subspace generated by $e_0$ and $U$ is the unilateral (forward) shift. Then $T$ is unitary and $T^{-1}$ is of the form $A + F$ with $A \in \mathcal{C} \cap \mathcal{C}^*$ and $F$ finite rank,

$$T^{-1} = \begin{pmatrix} U^* & 0 \\ e_0 \otimes e_0 & U \end{pmatrix}.$$

Clearly, $T^{-1} \notin \mathcal{C}$. 

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