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INVERTIBILITY IN NEST ALGEBRAS

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ABSTRACT. Let $\mathcal{F}$ denote a complete nest of subspaces of a complex Hilbert space $\mathcal{H}$, and let $\mathcal{C}$ denote the nest algebra defined by $\mathcal{F}$. Let $\mathcal{K}$ denote the ideal of compact operators on $\mathcal{H}$. If $\mathcal{F}$ has no infinite-dimensional gaps then $T \in \mathcal{C}$ is invertible in $\mathcal{C}$ if and only if it is invertible in $\mathcal{C} + \mathcal{K}$. An example is given of a nest with an infinite gap for which there exists an operator in $\mathcal{C}$ which is invertible in $\mathcal{C} + \mathcal{K}$ but not in $\mathcal{C}$.

1. Introduction. The idea of looking at a chain of invariant subspaces of an operator and relating it to the structure of the operator and its spectrum, is basic to finite-dimensional linear algebra. The first attempt to generalize such an approach for nonselfadjoint operators in Hilbert space seems to have been in a classical paper of M. Lifshitz [**6**]. This was later pursued by a number of mathematicians in the Soviet Union and is, for the most part, summarized in [**5**].

Independently, J. Ringrose [**7**] defined the notion of a nest algebra, an algebra of operators which leaves a chain of subspaces invariant, and studied the basic properties of such algebras. We are concerned with the question of identifying the invertible elements of a nest algebra $\mathcal{C}$. While it is always of interest to identify the invertible elements of a Banach algebra, in this case the problem has significance in the stability theory of input-output systems. The interested reader is referred to [**3, 4**].

It was shown in [**2**] that if $\mathcal{K}$ represents the ideal of compact operators on $\mathcal{H}$ and $\mathcal{C}$ is a nest algebra, then $\mathcal{C} + \mathcal{K}$ is a Banach algebra.

Here we show, that if a complete nest $\mathcal{F}$ has only finite-dimensional gaps and $\mathcal{C}$ is the nest algebra determined by $\mathcal{F}$, then $T \in \mathcal{C}$, where $T^{-1} \in \mathcal{C} + \mathcal{K}$ implies $T^{-1} \in \mathcal{C}$. This corrects the proof of and extends a previously announced result in this direction [**8**].

2. Preliminaries. Let $\mathcal{H}$ be a separable Hilbert space. A family of subspaces $\mathcal{F}$ of $\mathcal{H}$ will be called a *nest* if it is totally ordered by inclusion. $\mathcal{F}$ is complete if:

(i) $\{0\}, \mathcal{H} \in \mathcal{F}$;

(ii) given any subnest $\mathcal{F}_0 \subset \mathcal{F}$, the subspaces $\bigcap\{L: L \in \mathcal{F}_0\}$ and $\bigvee\{L: L \in \mathcal{F}_0\}$ are both members of $\mathcal{F}$.

Given a complete nest $\mathcal{F}$ and a nonzero subspace M in $\mathcal{F}$, we define the predecessor M_- of M by

$$M_- = \bigvee\{L: L \in \mathcal{F},\ L \subset M,\ L \neq M\}.$$

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If $\mathcal{C}$ is the family of operators defined by $\mathcal{C} = \{A \in \mathcal{B}(\mathcal{H}) : AL \subseteq L, \text{ for all } L \in \mathcal{F}\}$, it is easily seen that $\mathcal{C}$ is a weakly closed algebra containing the identity. $\mathcal{C}$ is called the nest algebra associated with $\mathcal{F}$.

Suppose $\mathcal{K}$ is the ideal of compact operators in $\mathcal{B}(\mathcal{H})$. The following is proved in [**2**].

THEOREM A. *$\mathcal{C} + \mathcal{K}$ is a norm closed algebra. The natural isomorphism of $\mathcal{C}/\mathcal{C} \cap \mathcal{K}$ and $\mathcal{C} + \mathcal{K}/\mathcal{C}$ is a Banach algebra isomorphism.*

3. The main result. Suppose $T \in \mathcal{C}$ is an invertible operator on $\mathcal{H}$ whose inverse is in $\mathcal{C} + \mathcal{K}$. Then $T^{-1} = A + K$ with $A \in \mathcal{C}$, $K \in \mathcal{K}$. Thus $I = TT^{-1} = TA + TK$, and $TK = I - TA$ is a compact operator in $\mathcal{C}$. If $\mathcal{F}$ is a continuous nest ($M_- = M$ for all $M \in \mathcal{F}$), it follows from a result of Gohberg-Krein [**5**] or Ringrose [**7**] that TK is quasinilpotent. Thus $(I - TK)$ is invertible and $(I - TK)^{-1}$ is in fact a power series in TK. It follows that $(I - TK)^{-1}$ and, therefore $T^{-1} = A(I - TK)^{-1}$, is in $\mathcal{C}$.

This argument does not work if $\mathcal{F}$ is not continuous, since $I - TK$ may not be invertible. We will show that if for all $M \in \mathcal{F}, \dim(M \ominus \mathcal{M}_-) < \infty$, then the same property holds. We begin with a preliminary lemma.

LEMMA 1. *Suppose T is an operator on $\mathcal{H}$ such that the restriction of T to an invariant subspace of finite codimension is invertible. If T has trivial kernel, then T is invertible on $\mathcal{H}$.*

PROOF. Suppose $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and T has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with respect to this decomposition. Then, it suffices to show that $T_3 = (I - P_\mathcal{M})T(I - P_\mathcal{M})$ is invertible on $\mathcal{M}^\perp$. Since $\mathcal{M}^\perp$ is finite dimensional we show T_3 has trivial kernel. If $x \in \mathcal{M}^\perp$ such that $T_3 x = 0$, this implies $Tx \in \mathcal{M}$. But then $Tx = T_1 y = Ty$ for some $y \in \mathcal{M}$. But this is impossible since T has trivial kernel and $x \in \mathcal{M}^\perp$.

THEOREM 2. *Let $\mathcal{F}$ be a complete nest on $\mathcal{H}$ with no infinite-dimensional gaps. If $\mathcal{C}$ is the nest algebra determined by $\mathcal{F}$ then $T \in \mathcal{C}$ and $T^{-1} \in \mathcal{C} + \mathcal{K}$ implies $T^{-1} \in \mathcal{C}$.*

PROOF. Suppose $\mathcal{M} \in \mathcal{F}$. Since, as above, $TA = I - TK$ implies $TK \in \mathcal{C}$ and TK is compact, it is enough to consider the case where $P_\mathcal{M}(I - TK)|\mathcal{M}$ is not invertible. For otherwise, $(I - TK)\mathcal{M} = \mathcal{M}$ implies $\mathcal{M} = TA\mathcal{M}$ and therefore $\mathcal{M} \supset A\mathcal{M} = T^{-1}\mathcal{M}$.

If $P_\mathcal{M}(I - TK)|\mathcal{M}$ is not invertible, then, since TK is compact, there exist at most finitely many $\{L_i\}_{i=1}^n \in \mathcal{F}$ with $L_i \neq L_{i-}$ such that $L_i \ominus L_{i-}$ contains a vector x_i with $(I - TK)x_i \in L_{i-}$. Assume $L_1 \subset L_2 \subset \cdots \subset L_n \subset \mathcal{M}$.

If $(L_1)_- = \{0\}$, then L_1 is a finite-dimensional subspace invariant under T and thus, by a simple dimension argument, $TL_1 = L_1$.

If $(L_1)_- \neq \{0\}$, then by the assumption on $\{L_i\}_{i=1}^n$, $P_{(L_1)_-}(I - TK)|(L_1)_-$ is invertible. By the argument of the first paragraph of the proof, we obtain $T(L_1)_- = (L_1)_-$. Applying Lemma 1 to T gives $TL_1 = L_1$.

Now consider L_2, which we write as $L_2 = (L_2 \ominus L_1) \oplus L_1$. Let $\hat{S}$ denote the compression of an operator $S \in \mathcal{C}$ to $L_1^\perp$. Since $TL_1 = L_1$, to show $TL_2 = L_2$ it suffices to show that $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$.

If $L_1 = (L_2)_-$, then $L_2 \ominus L_1$ is finite dimensional. Noting that T is invertible and $TL_1 = L_1$ implies $\hat{T}$ is invertible, and that $\hat{T}(L_2 \ominus L_1) \subset (L_2 \ominus L_1)$, it follows that $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$.

If $L_1 \neq (L_2)_-$, then $\hat{P}_{(L_2)_-}(\hat{I} - \hat{T}\hat{K})|[(L_2)_- \ominus L_1]$ is invertible. Thus by the argument given in the first paragraph of the proof, $\hat{T}[(L_2)_- \ominus L_1] = [(L_2)_- \ominus L_1]$. By Lemma 1, applied to $\hat{T}$ we obtain $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$ and thus $TL_2 = L_2$.

Since $\mathcal{M} = (\mathcal{M} \ominus L_n) \oplus (L_n \ominus L_{n-1}) \oplus \cdots \oplus (L_2 \ominus L_1) \oplus L_1$, the above argument applied n times gives $T\mathcal{M} = \mathcal{M}$ and completes the proof.

What happens when $\mathcal{F}$ has infinite-dimensional gaps? Before we answer this we give a result to show that this question for compact operators is equivalent to the question for finite rank operators. This may be of independent interest.

THEOREM 3. *Suppose that for $T \in \mathcal{C}$, $T^{-1} = A + F$, F finite rank, implies $F \in \mathcal{C}$. Then $T \in \mathcal{C}$ and $T^{-1} = A + K$, K compact, implies $K \in \mathcal{C}$.*

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PROOF. Suppose $T \in \mathcal{C}$ with $T^{-1} = A + K$. We show that for every $\varepsilon > 0$, where $\varepsilon < 1/\|T\|$, there exists an invertible operator $B_\varepsilon^{-1} = L + F$ with $L \in \mathcal{C}$, F finite rank, $B_\varepsilon \in \mathcal{C}$ and $\|B_\varepsilon^{-1} - T^{-1}\| < \varepsilon$. Since, by hypothesis, $B_\varepsilon^{-1} \in \mathcal{C}$, this will imply $T^{-1} \in \mathcal{C}$.

If $\|B_\varepsilon^{-1} - T^{-1}\| < 1/\|T\|$, it follows that $B_\varepsilon = T\sum_{i=0}^{\infty}(I - B_\varepsilon^{-1}T)^i$. Thus if $B_\varepsilon^{-1}T \in \mathcal{C}$ so is B_ε. As we have seen, $T^{-1} = A + K$ implies $KT \in \mathcal{C}$. Thus [**1**] there exists a sequence $\{F_n\}$ of finite rank operators in $\mathcal{C}$ such that $\|F_n - KT\| \to 0$ as $n \to \infty$.

Let $S_n = A + F_nT^{-1}$. Then $S_n - T^{-1} = F_nT^{-1} - K$ and, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$, $\|S_n - T^{-1}\| < \varepsilon$. If $\varepsilon < 1/(|T|)$, it follows that S_n is invertible and

$$S_nT = AT + F_nT^{-1}T = AT + F_n \in \mathcal{C}.$$

Thus given $\varepsilon > 0$, choose $B_\varepsilon^{-1} = S_n$ for $n > N(\varepsilon)$. This completes the proof.

EXAMPLE 1. Let $\mathcal{H} = l^2(0,\infty) \oplus l^2(0,\infty)$ with the nest whose projections are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} U & e_0 \otimes e_0 \\ 0 & U^* \end{pmatrix}$$

where $e_0 \otimes e_0$ is the projection on the one-dimensional subspace generated by e_0 and U is the unilateral (forward) shift. Then T is unitary and T^{-1} is of the form $A + F$ with $A \in \mathcal{C} \cap \mathcal{C}^*$ and F finite rank,

$$T^{-1} = \begin{pmatrix} U^* & 0 \\ e_0 \otimes e_0 & U \end{pmatrix}.$$

Clearly, $T^{-1} \notin \mathcal{C}$.

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