

INVERTIBILITY IN NEST ALGEBRAS

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ABSTRACT. Let \mathcal{F} denote a complete nest of subspaces of a complex Hilbert space \mathcal{H} , and let \mathcal{C} denote the nest algebra defined by \mathcal{F} . Let \mathcal{K} denote the ideal of compact operators on \mathcal{H} . If \mathcal{F} has no infinite-dimensional gaps then $T \in \mathcal{C}$ is invertible in \mathcal{C} if and only if it is invertible in $\mathcal{C} + \mathcal{K}$. An example is given of a nest with an infinite gap for which there exists an operator in \mathcal{C} which is invertible in $\mathcal{C} + \mathcal{K}$ but not in \mathcal{C} .

1. Introduction. The idea of looking at a chain of invariant subspaces of an operator and relating it to the structure of the operator and its spectrum, is basic to finite-dimensional linear algebra. The first attempt to generalize such an approach for nonselfadjoint operators in Hilbert space seems to have been in a classical paper of M. Lifshitz [6]. This was later pursued by a number of mathematicians in the Soviet Union and is, for the most part, summarized in [5].

Independently, J. Ringrose [7] defined the notion of a nest algebra, an algebra of operators which leaves a chain of subspaces invariant, and studied the basic properties of such algebras. We are concerned with the question of identifying the invertible elements of a nest algebra \mathcal{C} . While it is always of interest to identify the invertible elements of a Banach algebra, in this case the problem has significance in the stability theory of input-output systems. The interested reader is referred to [3, 4].

It was shown in [2] that if \mathcal{K} represents the ideal of compact operators on \mathcal{H} and \mathcal{C} is a nest algebra, then $\mathcal{C} + \mathcal{K}$ is a Banach algebra.

Here we show, that if a complete nest \mathcal{F} has only finite-dimensional gaps and \mathcal{C} is the nest algebra determined by \mathcal{F} , then $T \in \mathcal{C}$, where $T^{-1} \in \mathcal{C} + \mathcal{K}$ implies $T^{-1} \in \mathcal{C}$. This corrects the proof of and extends a previously announced result in this direction [8].

2. Preliminaries. Let \mathcal{H} be a separable Hilbert space. A family of subspaces \mathcal{F} of \mathcal{H} will be called a *nest* if it is totally ordered by inclusion. \mathcal{F} is complete if:

- (i) $\{0\}, \mathcal{H} \in \mathcal{F}$;
- (ii) given any subnest $\mathcal{F}_0 \subset \mathcal{F}$, the subspaces $\bigcap\{L: L \in \mathcal{F}_0\}$ and $\bigvee\{L: L \in \mathcal{F}_0\}$ are both members of \mathcal{F} .

Given a complete nest \mathcal{F} and a nonzero subspace M in \mathcal{F} , we define the predecessor M_- of M by

$$M_- = \bigvee\{L: L \in \mathcal{F}, L \subset M, L \neq M\}.$$

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If \mathcal{C} is the family of operators defined by $\mathcal{C} = \{A \in \mathcal{B}(\mathcal{X}) : AL \subseteq L, \text{ for all } L \in \mathcal{F}\}$, it is easily seen that \mathcal{C} is a weakly closed algebra containing the identity. \mathcal{C} is called the nest algebra associated with \mathcal{F} .

Suppose \mathcal{K} is the ideal of compact operators in $\mathcal{B}(\mathcal{X})$. The following is proved in [2].

THEOREM A. $\mathcal{C} + \mathcal{K}$ is a norm closed algebra. The natural isomorphism of $\mathcal{C}/\mathcal{C} \cap \mathcal{K}$ and $\mathcal{C} + \mathcal{K}/\mathcal{C}$ is a Banach algebra isomorphism.

3. The main result. Suppose $T \in \mathcal{C}$ is an invertible operator on \mathcal{X} whose inverse is in $\mathcal{C} + \mathcal{K}$. Then $T^{-1} = A + K$ with $A \in \mathcal{C}$, $K \in \mathcal{K}$. Thus $I = TT^{-1} = TA + TK$, and $TK = I - TA$ is a compact operator in \mathcal{C} . If \mathcal{F} is a continuous nest ($M_- = M$ for all $M \in \mathcal{F}$), it follows from a result of Gohberg-Krein [5] or Ringrose [7] that TK is quasinilpotent. Thus $(I - TK)$ is invertible and $(I - TK)^{-1}$ is in fact a power series in TK . It follows that $(I - TK)^{-1}$ and, therefore $T^{-1} = A(I - TK)^{-1}$, is in \mathcal{C} .

This argument does not work if \mathcal{F} is not continuous, since $I - TK$ may not be invertible. We will show that if for all $M \in \mathcal{F}$, $\dim(M \ominus M_-) < \infty$, then the same property holds. We begin with a preliminary lemma.

LEMMA 1. Suppose T is an operator on \mathcal{X} such that the restriction of T to an invariant subspace of finite codimension is invertible. If T has trivial kernel, then T is invertible on \mathcal{X} .

PROOF. Suppose $\mathcal{X} = M \oplus M^\perp$ and T has the matrix representation

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

with respect to this decomposition. Then, it suffices to show that $T_3 = (I - P_M)T(I - P_M)$ is invertible on M^\perp . Since M^\perp is finite dimensional we show T_3 has trivial kernel. If $x \in M^\perp$ such that $T_3x = 0$, this implies $Tx \in M$. But then $Tx = T_1y = Ty$ for some $y \in M$. But this is impossible since T has trivial kernel and $x \in M^\perp$.

THEOREM 2. Let \mathcal{F} be a complete nest on \mathcal{X} with no infinite-dimensional gaps. If \mathcal{C} is the nest algebra determined by \mathcal{F} then $T \in \mathcal{C}$ and $T^{-1} \in \mathcal{C} + \mathcal{K}$ implies $T^{-1} \in \mathcal{C}$.

PROOF. Suppose $M \in \mathcal{F}$. Since, as above, $TA = I - TK$ implies $TK \in \mathcal{C}$ and TK is compact, it is enough to consider the case where $P_M(I - TK)|_M$ is not invertible. For otherwise, $(I - TK)M = M$ implies $M = TAM$ and therefore $M \supset AM = T^{-1}M$.

If $P_M(I - TK)|_M$ is not invertible, then, since TK is compact, there exist at most finitely many $\{L_i\}_{i=1}^n \in \mathcal{F}$ with $L_i \neq L_{i-}$ such that $L_i \ominus L_{i-}$ contains a vector x_i with $(I - TK)x_i \in L_{i-}$. Assume $L_1 \subset L_2 \subset \dots \subset L_n \subset M$.

If $(L_1)_- = \{0\}$, then L_1 is a finite-dimensional subspace invariant under T and thus, by a simple dimension argument, $TL_1 = L_1$.

If $(L_1)_- \neq \{0\}$, then by the assumption on $\{L_i\}_{i=1}^n$, $P_{(L_1)_-}(I - TK)|(L_1)_-$ is invertible. By the argument of the first paragraph of the proof, we obtain $T(L_1)_- = (L_1)_-$. Applying Lemma 1 to T gives $TL_1 = L_1$.

Now consider L_2 , which we write as $L_2 = (L_2 \ominus L_1) \oplus L_1$. Let \hat{S} denote the compression of an operator $S \in \mathcal{C}$ to L_1^\perp . Since $TL_1 = L_1$, to show $TL_2 = L_2$ it suffices to show that $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$.

If $L_1 = (L_2)_-$, then $L_2 \ominus L_1$ is finite dimensional. Noting that T is invertible and $TL_1 = L_1$ implies \hat{T} is invertible, and that $\hat{T}(L_2 \ominus L_1) \subset (L_2 \ominus L_1)$, it follows that $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$.

If $L_1 \neq (L_2)_-$, then $\hat{P}_{(L_2)_-}(\hat{I} - \hat{T}\hat{K})[(L_2)_- \ominus L_1]$ is invertible. Thus by the argument given in the first paragraph of the proof, $\hat{T}[(L_2)_- \ominus L_1] = [(L_2)_- \ominus L_1]$. By Lemma 1, applied to \hat{T} we obtain $\hat{T}(L_2 \ominus L_1) = L_2 \ominus L_1$ and thus $TL_2 = L_2$.

Since $\mathcal{M} = (\mathcal{M} \ominus L_n) \oplus (L_n \ominus L_{n-1}) \oplus \dots \oplus (L_2 \ominus L_1) \oplus L_1$, the above argument applied n times gives $T\mathcal{M} = \mathcal{M}$ and completes the proof.

What happens when \mathcal{F} has infinite-dimensional gaps? Before we answer this we give a result to show that this question for compact operators is equivalent to the question for finite rank operators. This may be of independent interest.

THEOREM 3. *Suppose that for $T \in \mathcal{C}$, $T^{-1} = A + F$, F finite rank, implies $F \in \mathcal{C}$. Then $T \in \mathcal{C}$ and $T^{-1} = A + K$, K compact, implies $K \in \mathcal{C}$.*

PROOF. Suppose $T \in \mathcal{C}$ with $T^{-1} = A + K$. We show that for every $\varepsilon > 0$, where $\varepsilon < 1/\|T\|$, there exists an invertible operator $B_\varepsilon^{-1} = L + F$ with $L \in \mathcal{C}$, F finite rank, $B_\varepsilon \in \mathcal{C}$ and $\|B_\varepsilon^{-1} - T^{-1}\| < \varepsilon$. Since, by hypothesis, $B_\varepsilon^{-1} \in \mathcal{C}$, this will imply $T^{-1} \in \mathcal{C}$.

If $\|B_\varepsilon^{-1} - T^{-1}\| < 1/\|T\|$, it follows that $B_\varepsilon = T \sum_{i=0}^\infty (I - B_\varepsilon^{-1}T)^i$. Thus if $B_\varepsilon^{-1}T \in \mathcal{C}$ so is B_ε . As we have seen, $T^{-1} = A + K$ implies $KT \in \mathcal{C}$. Thus [1] there exists a sequence $\{F_n\}$ of finite rank operators in \mathcal{C} such that $\|F_n - KT\| \rightarrow 0$ as $n \rightarrow \infty$.

Let $S_n = A + F_nT^{-1}$. Then $S_n - T^{-1} = F_nT^{-1} - K$ and, given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$, $\|S_n - T^{-1}\| < \varepsilon$. If $\varepsilon < 1/(\|T\|)$, it follows that S_n is invertible and

$$S_nT = AT + F_nT^{-1}T = AT + F_n \in \mathcal{C}.$$

Thus given $\varepsilon > 0$, choose $B_\varepsilon^{-1} = S_n$ for $n > N(\varepsilon)$. This completes the proof.

EXAMPLE 1. Let $\mathcal{X} = l^2(0, \infty) \oplus l^2(0, \infty)$ with the nest whose projections are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} U & e_0 \otimes e_0 \\ 0 & U^* \end{pmatrix}$$

where $e_0 \otimes e_0$ is the projection on the one-dimensional subspace generated by e_0 and U is the unilateral (forward) shift. Then T is unitary and T^{-1} is of the form $A + F$ with $A \in \mathcal{C} \cap \mathcal{C}^*$ and F finite rank,

$$T^{-1} = \begin{pmatrix} U^* & 0 \\ e_0 \otimes e_0 & U \end{pmatrix}.$$

Clearly, $T^{-1} \notin \mathcal{C}$.

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