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ON A GENERALIZED MOMENT PROBLEM. II

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ABSTRACT. Recently, we have extended the well-known Müntz-Szász theorem by showing that if $f(z)$ is absolutely continuous and $|f'(x)| \geqslant k > 0$ a.e. on (a, b), where $a \geqslant 0$ and if $\{n_p\}$ is a sequence of positive numbers tending to infinity and satisfying $\sum_{p=1}^{\infty} 1/n_p = \infty$, then the sequence $\{f(x)^{n_p}\}$ is complete on (a, b) if and only if $f(x)$ is strictly monotone on (a, b). We now apply Zarecki's theorem to improve the condition "$|f'(x)| \geqslant k > 0$ a.e. on (a, b)" by the condition "$f'(x) \neq 0$ a.e. on (a, b)". Furthermore, we extend some well-known theorems of Picone, Mikusiński, and Boas.

1. Introduction. Let $L(a, b)$ be the space of all summable functions defined on the finite interval (a, b). As usual (see R. P. Boas [**2**, p. 234]) a sequence of functions $f_n(x)$ is complete on (a, b) if for any $g \in L(a, b)$, the equalities

$$\int_a^b f_n(x) g(x)\, dx = 0, \qquad n = 1, 2, \ldots,$$

imply that $g(x) = 0$ a.e. (almost everywhere) on (a, b).

Recently, in [**3**], we have proved the following extension of the Müntz-Szász theorem (see Boas [**2**, p. 235]).

THEOREM 1. *Let $\{n_p\}$ be a sequence of positive numbers tending to infinity and satisfying*

$$\sum_{p=1}^{\infty} \frac{1}{n_p} = \infty. \tag{1}$$

Let $f(x)$ be a function absolutely continuous on (a, b) with $f(a)f(b) \geqslant 0$, and let its derivative satisfy $|f'(x)| \geqslant k > 0$ a.e. on (a, b). Then the sequence $\{f(x)^{n_p}\}$ is complete on (a, b) if and only if the function $f(x)$ is monotone on (a, b).

In this note, we shall improve Theorem 1 as follows.

THEOREM 2. *The assertion of Theorem* 1 *can be extended by replacing the condition* "$|f'(x)| \geqslant k > 0$ *a.e. on* (a, b)" *by the condition* "$f'(x) \neq 0$ *a.e. on* (a, b)".

2. Inverse function. To prove Theorem 2, we shall first study the inverse of a monotone function $f(x)$ on (a, b), where monotone function means either strictly increasing or decreasing on (a, b). As usual, we let $f^{-1}(x)$ be the inverse of $f(x)$. We

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shall need the following theorem of Zarecki (see Natanson [**5**, I, p. 271] or Saks [**6**, p. 128]).

LEMMA 1. *Let $f(x)$ be a monotone function on (a, b). Then a necessary and sufficient condition that the inverse $f^{-1}(x)$ be absolutely continuous is that $f'(x) \neq 0$ a.e. on (a, b).*

We note that the hypothesis "$f'(x) \neq 0$ a.e. on (a, b)" in Theorem 2 serves to ensure that the inverse $f^{-1}(x)$ is absolutely continuous on (a, b).

3. Proof of Theorem 2. If $f(x)$ is not monotone on (a, b), then the sequence $\{f(x)^{n_p}\}$ is incomplete on (a, b) due to [**3**].

Conversely, let $f(x)$ be, say, strictly increasing on (a, b). We shall prove that the sequence $\{f(x)^{n_p}\}$ is complete on (a, b). For this, we consider an arbitrary function $g \in L(a, b)$ such that

$$\int_a^b f(x)^{n_p} g(x)\, dx = 0, \qquad p = 1, 2, \ldots. \tag{2}$$

Since $f'(x) \neq 0$ a.e. on (a, b), it follows from Lemma 1 that the inverse $f^{-1}(x)$ is absolutely continuous on $(f(a), f(b))$. This allows us to change from a Stieltjes to a Lebesgue integral (see [**5**, I, p. 265 and II, p. 236]). For this, we write $y = f(x)$ and $x = f^{-1}(y)$. Then by (2), we obtain

$$\int_{f(a)}^{f(b)} y^{n_p} g(f^{-1}(y))(f^{-1}(y))'\, dy = 0, \qquad p = 1, 2, \ldots. \tag{3}$$

Since the function $g(f^{-1}(y))(f^{-1}(y))'$ is summable on $(f(a), f(b))$, it follows from (3) and the Müntz-Szász theorem that

$$g(f^{-1}(y))(f^{-1}(y))' = 0 \quad \text{a.e. on } (f(a), f(b)).$$

The function $f(x)$ is absolutely continuous on (a, b) so by Lemma 1, we can see that the derivative of the inverse function $(f^{-1}(y))' \neq 0$ a.e. on $(f(a), f(b))$. This yields $g(x) = 0$ a.e. on (a, b), and therefore the sequence $\{f(x)^{n_p}\}$ is complete on (a, b). This completes the proof.

4. Picone's theorem. Early in the fifties, Boas [**1**] proved the following extension of Picone's theorem, which also extended to that of Mikusiński [**4**].

THEOREM B. *Let $\{n_p\}$ be a sequence of positive numbers satisfying* (1) *and $n_q - n_p > (q - p)d$, where $d > 0$ and $p < q$. If $g \in L(a, b)$, where $a \geqslant 0$, and if for some c with $a < c < b$,*

$$\int_a^b x^{n_p} g(x)\, dx = O(c^{n_p}), \qquad p = 1, 2, \ldots,$$

then $g(x) = 0$ a.e. on (a, b).

In fact, Boas considered a sequence of complex numbers n_p with additional assumptions. For simplicity, we consider only the positive real case. We shall extend the above uniqueness theorem of Boas as follows.

THEOREM 3. *Let $f(x)$ be a function absolutely continuous and monotonically increasing on (a, b) whose derivative $f'(x) \neq 0$ a.e. on (a, b), where $f(a) \geqslant 0$. Let $\{n_p\}$ be a sequence of positive numbers satisfying* (1) *and $n_q - n_p > (q-p)d$, where $d > 0$ and $p < q$. If $g \in L(a, b)$ and if for some c with $f(a) < c < f(b)$,*

$$\int_a^b f(x)^{n_p} g(x)\, dx = O(c^{n_p}), \qquad p = 1, 2, \ldots, \tag{4}$$

then $g(x) = 0$ a.e. on (a, b).

PROOF. As before, we let $y = f(x)$ and $x = f^{-1}(y)$. Then equation (4) becomes

$$\int_{f(a)}^{f(b)} y^{n_p} g(f^{-1}(y))(f^{-1}(y))'\, dy = O(c^{n_p}).$$

Since the function $g(f^{-1}(y))(f^{-1}(y))'$ is summable on $(f(a), f(b))$, it follows from Theorem B that

$$g(f^{-1}(y))(f^{-1}(y))' = 0 \quad \text{a.e. on } (f(a), f(b)).$$

As before, we know that the function $(f^{-1}(y))' \neq 0$ a.e. on $(f(a), f(b))$. This yields the assertion $g(x) = 0$ a.e. on (a, b).

We shall now consider the following related theorem of Boas [**1**] about the asymptotic behaviour of moments.

THEOREM B*. *If $g(x)$ is a function in $L(a, b)$, where $a \geqslant 0$, and does not vanish almost everywhere in a neighborhood of b, then*

$$\limsup_{n \to \infty} \left| \int_a^b x^n g(x)\, dx \right|^{1/n} = b.$$

By the same argument as in Theorem 3 together with Theorem B*, we obtain the following asymptotic behaviour of generalized moments.

THEOREM 4. *Let $f(x)$ be a function absolutely continuous and monotonically increasing on (a, b) whose derivative $f'(x) \neq 0$ a.e. on (a, b), where $f(a) \geqslant 0$. If $g \in L(a, b)$ and does not vanish a.e. in a neighborhood of b, then*

$$\limsup_{n \to \infty} \left| \int_a^b f(x)^n g(x)\, dx \right|^{1/n} = f(b).$$

The conditions of $f(x)$ in the above theorems could be improved. To discuss this, we shall now prove the following extension of Boas' theorem [**2**, p. 233].

THEOREM 5. *Let $f(x)$ be a nonnegative function monotonically increasing on (a, b) and continuous in a neighborhood of b. If $g(x)$ is summable and nonnegative on (a, b), and does not vanish a.e. in a neighborhood of b, then*

$$\lim_{n \to \infty} \left\{ \int_a^b f(x)^n g(x)\, dx \right\}^{1/n} = f(b).$$

PROOF. For each sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}(f(b-\varepsilon))^n \int_{b-\varepsilon}^b g(x)\, dx &\leqslant \int_{b-\varepsilon}^b f(x)^n g(x)\, dx \\ &\leqslant \int_a^b f(x)^n g(x)\, dx \leqslant f(b)^n \int_a^b g(x)\, dx.\end{aligned}$$

By taking the nth roots and applying the hypothesis of $g(x)$, we obtain

$$f(b-\varepsilon) \leqslant \varliminf_{n\to\infty} \left\{ \int_a^b f(x)^n g(x)\,dx \right\}^{1/n} \leqslant \varlimsup_{n\to\infty} \left\{ \int_a^b f(x)^n g(x)\,dx \right\}^{1/n} \leqslant f(b).$$

The assertion now follows from the continuity of $f(x)$ in a neighborhood of b.

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