PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 91, Number 4, August 1984

COMPLEX FOLIATIONS GENERATED BY (1, 1)-FORMS

M. KLIMEK[1]

ABSTRACT. Complex foliations generated by (1, 1)-forms are studied in order to describe geometric properties of complex partial differential equations of Monge-Ampère type.

1. Introduction. Let $\alpha = (i/2)\Sigma\alpha_{jk}\, dz_j \wedge d\bar{z}_k$ be a $\mathscr{C}^1$-differential form on an open set $\Omega \subset \mathbf{C}^n$. By α^m we shall denote the mth exterior power of α ($\alpha^0 \equiv 1$). If $z \in \Omega$ then $N_z(\alpha)$ is the *nullity space* of α at z i.e.

$$N_z(\alpha) = \{x \in \mathbf{C}^n : \alpha_z(x, y) = 0 \text{ for all } y \in \mathbf{C}^n\}.$$

If for each $z \in \Omega$, the linear operator corresponding to the matrix $[\alpha_{jk}(z)]$ is reduced by its range we say that α is *reducible*. A detailed geometric characterization of the notion is given in the third section. In particular it is proved there that if the complex rank of α is p then the mapping $z \to N_z(\alpha)$ is a $\mathscr{C}^1$-distribution of real dimension $2(n-p)$ if and only if α is reducible.

The aim of this paper is to study complex foliations (i.e. $\mathscr{C}^1$-foliations whose leaves are complex manifolds) generated by reducible differential forms. We are particularly interested in forms $dd^c u$ where u is a complex valued $\mathscr{C}^3$-function satisfying the Monge-Ampère equation

$$(dd^c u)^{p+1} = 0 \tag{1}$$

in an open set $\Omega \subset \mathbf{C}^n$ with $0 < p < n$. (Recall that $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$.) It is known (see [**1**]) that if the above equation is nondegenerate i.e.

$$(dd^c u)^p \neq 0 \tag{2}$$

in Ω and $\operatorname{Im} u$ is plurisubharmonic then there exists a complex foliation $\mathscr{F}$ of Ω such that u is pluriharmonic along the leaves of $\mathscr{F}$. We shall prove that this remains true in a more general case when $dd^c u$ is reducible. Furthermore if for a complex function u of class $\mathscr{C}^3$ the nullity spaces of $dd^c u$ generate a complex foliation of codimension p then $dd^c u$ must be reducible.

We shall examine the system of equations of Monge-Ampère type

$$(dd^c u)^p \wedge (dd^c v)^{q+1} = 0, \tag{3}$$

$$(dd^c u)^{p+1} = 0 \tag{4}$$

Received by the editors October 11, 1983.
1980 *Mathematics Subject Classification*. Primary 32F99; Secondary 32C25, 31C10, 32A10.
[1] The author gratefully acknowledges support received from the Irish Department of Education.

with the additional nondegeneracy condition

$$(dd^c u)^p \wedge (dd^c v)^q \neq 0 \tag{5}$$

where p and q are nonnegative integers, $0 < p + q < n$, u and v are complex functions on an open subset Ω of $\mathbf{C}^n$. The conditions (4) and (5) imply that the rank of $dd^c u$ is constant. Nevertheless the rank of $dd^c v$ may vary. In particular the equation $(dd^c v)^{p+q+1} = 0$ may be degenerate (cf. Example 5). In general, very little is known about such equations (see [**1, 2**]). We shall prove that if $dd^c u$ and $dd^c v$ are reducible $\mathscr{C}^1$-forms satisfying (3), (4) and (5) then there is a complex foliation $\mathscr{F}$ of Ω such that both u and v are pluriharmonic when restricted to any leaf of $\mathscr{F}$.

2. Main results. We shall prove the following.

THEOREM 1. *Let $\alpha = (i/2)\Sigma\alpha_{jk}\, dz_j \wedge d\bar{z}_k$ and $\beta = (i/2)\Sigma\beta_{jk}\, dz_j \wedge d\bar{z}_k$ be reducible $\mathscr{C}^1$-forms on an open subset Ω of $\mathbf{C}^n$. Let $A = [\alpha_{jk}]^{\mathrm{tr}}$ and $B = [\beta_{jk}]^{\mathrm{tr}}$. Assume p and q are nonnegative integers such that $0 < p + q < n$ and the conditions*

$$\alpha^p \wedge \beta^q \neq 0, \quad \alpha^p \wedge \beta^{q+1} = 0, \quad \alpha^{p+1} = 0$$

are satisfied in Ω. Then $\Delta(z) = N_z(\alpha) \cap N_z(\beta)$ is a $\mathscr{C}^1$-distribution over Ω of (real) dimension $2(n - p - q)$ and $\Delta(z) = \operatorname{Ker} A(z) \cap \operatorname{Ker} B(z)$ for all $z \in \Omega$. If $d\alpha$ and $d\beta$ vanish on $\Delta(z) \times \Delta(z) \times \mathbf{C}^n$ for each $z \in \Omega$ the distribution Δ is integrable and generates a complex foliation $\mathscr{F}$ of Ω of complex codimension $p + q$.

The above theorem generalizes some earlier results obtained by Bedford, Kalka [**1**] and Kalina [**3**]. Under the additional assumption that $\alpha = \beta$ and $\operatorname{Im}\alpha \geqslant 0$ it has been proved in [**1**]. For real forms α of class $\mathscr{C}^3$ and $\beta = dd^c u$ with $\operatorname{Im} u$ plurisubharmonic the result has been shown by Kalina [**3**]. However the way in which we prove Theorem 1 in this paper is different from the methods used in [**1** and **3**].

Theorem 1 yields the following two results.

THEOREM 2. *Let u and v be complex functions of class $\mathscr{C}^3$ on $\Omega \subset \mathbf{C}^n$ such that $dd^c u$ and $dd^c v$ are reducible. Let p and q be nonnegative integers such that $0 < p + q < n$ and the conditions* (3), (4) *and* (5) *are satisfied in Ω. Then there exists a complex foliation $\mathscr{F}$ of Ω by complex submanifolds of codimension $p + q$ such that for any leaf M of $\mathscr{F}$, the restrictions of u and v to M are pluriharmonic on M and $\partial \operatorname{Re} u/\partial z_j$, $\partial \operatorname{Im} u/\partial z_j$, $\partial \operatorname{Re} v/\partial z_j$, $\partial \operatorname{Im} v/\partial z_j$ are holomorphic on M for $j = 1, 2, \ldots, n$.*

THEOREM 3. *Suppose u is a complex $\mathscr{C}^3$-function on Ω such that $\partial\bar{\partial} u$ is reducible and for some p $(0 < p < n)$*

$$\partial\bar{u} \wedge \bar{\partial} u \wedge (\partial\bar{\partial} u)^p \neq 0, \quad \partial\bar{u} \wedge \bar{\partial} u \wedge (\partial\bar{\partial} u)^{p+1} = 0$$

in Ω. Then there is a complex foliation $\mathscr{F}$ on Ω of (complex) codimension $p + 1$ such that for every leaf M of $\mathscr{F}$, $u|M$ is holomorphic on M.

If $u = v$ and $\operatorname{Im} u$ is plurisubharmonic Theorems 2 and 3 reduce to Theorems 2.4 and 5.1 in [**1**] respectively.

3. Reducible forms. For any $\mathbf{C}$-linear operator $A\colon \mathbf{C}^n \to \mathbf{C}^n$, $\operatorname{Ker} A$ and $\operatorname{Ran} A$ denote the kernel and range of A respectively. We consider $\mathbf{C}^n$ as a Hilbert space

with the inner product $\langle z, w\rangle = \Sigma z_j \bar{w}_j$. Every operator A induces the orthogonal decompositions $\mathbf{C}^n = \operatorname{Ran} A \oplus \operatorname{Ker} A^* = \operatorname{Ran} A^* \oplus \operatorname{Ker} A$. Moreover if U is a unitary operator and P is a projection then UPU^* is again a projection. Hence we have

LEMMA 1. *If A: $\mathbf{C}^n \to \mathbf{C}^n$ is a $\mathbf{C}$-linear operator of rank $p < n$ then the following conditions are equivalent*:

(1) $\operatorname{Ran} A$ *reduces* A,

(2) $\operatorname{Ran} A = \operatorname{Ran} A^*$ *and* $\operatorname{Ker} A = \operatorname{Ker} A^*$,

(3) *A commutes with the orthogonal projection onto* $\operatorname{Ran} A$

(4) *there is a unitary operator U: $\mathbf{C}^n \to \mathbf{C}^n$ and a $\mathbf{C}$-linear isomorphism B: $\mathbf{C}^p \to \mathbf{C}^p$ such that for all $(z, w) \in \mathbf{C}^p \times \mathbf{C}^{n-p}$*

$$(B(z), 0) = UAU^*(z, w).$$

If A satisfies any of the above conditions, we will say it is *reducible*.

EXAMPLE 1. If A is reducible then A^{tr} is also reducible.

EXAMPLE 2. If $A = S + iT$ where both S and T are Hermitian operators and either S or T is semidefinite then A is reducible. To see this observe if $z \in \operatorname{Ker} A$ then $\langle Az, z\rangle = 0$ and hence $\langle Sz, z\rangle = \langle Tz, z\rangle = 0$ since S and T are Hermitian. If —for instance—S is positive semidefinite, it has a square root $S^{1/2}$, so that $\langle S^{1/2}z, S^{1/2}z\rangle = 0$ and hence $z \in \operatorname{Ker} S$. Consequently $z \in \operatorname{Ker} T$. Therefore $z \in \operatorname{Ker} A^*$.

EXAMPLE 3. Suppose A is normal i.e. it commutes with its conjugate. Since eigenvectors of A belonging to different eigenvalues are orthogonal, A can be diagonalized by a unitary transformation and so it is reducible.

LEMMA 2. *If A: $\mathbf{C}^n \to \mathbf{C}^n$ is $\mathbf{C}$-linear then $N = \{z \in \mathbf{C}^n$: $\langle Az, w\rangle = \langle Aw, z\rangle$ for all $w \in \mathbf{C}^n\} = \operatorname{Ker} A \cap \operatorname{Ker} A^*$.*

PROOF. Set $S = \frac{1}{2}(A + A^*)$ and $T = (A - A^*)/2i$. Then both S and T are selfadjoint and $A = S + iT$. Clearly $\operatorname{Ker} A \cap \operatorname{Ker} A^* \subset N$. To prove the opposite inclusion it is enough to show that $N \subset \operatorname{Ker} S \cap \operatorname{Ker} T$. Fix $z \in N$. Then for all $w \in \mathbf{C}^n$

$$(\langle Sz, w\rangle - \langle Sw, z\rangle) + i(\langle Tz, w\rangle - \langle Tw, z\rangle) = 0.$$

But $\langle Sz, w\rangle - \langle Sw, z\rangle = 2i \operatorname{Im}\langle Sz, w\rangle$ and $\langle Tz, w\rangle - \langle Tw, z\rangle = 2i \operatorname{Im}\langle Tz, w\rangle$. Moreover $\langle x, y\rangle = \operatorname{Im}\langle x, -iy\rangle + i \operatorname{Im}\langle x, y\rangle$ for all $x, y \in \mathbf{C}^n$. Thus for all $w \in \mathbf{C}^n$, $\langle Sz, w\rangle = \langle Tz, w\rangle = 0$ and we are done.

COROLLARY 1. $N = (\operatorname{Ran} A + \operatorname{Ran} A^*)^{\perp}$.

LEMMA 3. *If $\alpha = (i/2)\Sigma\alpha_{jk}\, dz_j \wedge d\bar{z}_k$ is a $(1,1)$-form with constant coefficients and $A = [\alpha_{jk}]^{\mathrm{tr}}$ then*

$$\alpha(x, y) = \frac{i}{2}(\langle Ax, y\rangle - \langle Ay, x\rangle).$$

PROOF. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then $dz_j \wedge d\bar{z}_k(x, y) = x_j\bar{y}_k - y_j\bar{x}_k$ which yields the lemma.

For a positive integer p define c_p by setting $c_p = 2^{-p}$ if p is even and $c_p = i2^{-p}$ if p is odd. Then for multi-indices $J = (j_1, \ldots, j_p)$ and $K = (k_1, \ldots, k_p)$ we have

$$\left(\frac{i}{2}\right)^p dz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_p} = c_p\, dz^J \wedge d\bar{z}^K.$$

As a consequence of this equality and the definition of the determinant function we get

LEMMA 4. *If* $\alpha = (i/2)\Sigma\alpha_{jk}\, dz_j \wedge d\bar{z}_k$ *has constant coefficients then*

$$\alpha^p = p!c_p \sum_{J,K}{}' \alpha_{J,K}\, dz^J \wedge d\bar{z}^K$$

where the summation is taken over all multi-indices $J = (j_1, \ldots, j_p)$ *and* $K = (k_1, \ldots, k_p)$ *such that* $1 \leqslant j_1 < \cdots < j_p \leqslant n$ *and* $1 \leqslant k_1 < \cdots < k_p \leqslant n$ *and* $\alpha_{J,K} = \det[\alpha_{j,k}]_{j \in J, k \in K}$.

Let Ω be an open subset of $\mathbf{C}^n$ and let $\alpha = (i/2)\Sigma\alpha_{jk}\, dz_j \wedge d\bar{z}_k$ be a $\mathscr{C}^1$-form which is reducible (i.e. $[\alpha_{jk}(z)]$ is reducible for all $z \in \Omega$). Then $A(z) = [\alpha_{jk}(z)]^{\mathrm{tr}}$ is reducible for each $z \in \Omega$ and the above lemmas imply

COROLLARY 2. *If* α *is a reducible* $\mathscr{C}^1$*-form on* Ω *and* $z \in \Omega$ *then the nullity space of* α *at* z *is a complex subspace of* $\mathbf{C}^n$. *Moreover*

$$N_z(\alpha) = (\operatorname{Ran} A(z))^{\perp} = \operatorname{Ker} A(z).$$

As a consequence of Corollaries 1 and 2 and Lemma 4 we obtain

COROLLARY 3. *Let* α *be a* $(1,1)$*-form of class* $\mathscr{C}^1$ *on an open set* $\Omega \subset \mathbf{C}^n$ *such that the complex rank of* α *is* p *(i.e.* $\alpha^p \neq 0$, $\alpha^{p+1} = 0$*). Then the mapping* $z \to N_z(\alpha)$ *is a* $\mathscr{C}^1$*-distribution of real dimension* $2(n-p)$ *if and only if* α *is reducible.*

Since $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$, $dd^c = 2i\partial\bar{\partial}$. Let u be a complex $\mathscr{C}^3$-function on Ω such that at least one of the functions $\operatorname{Re} u$, $-\operatorname{Re} u$, $\operatorname{Im} u$, $-\operatorname{Im} u$ is plurisubharmonic. Then—in view of Example 2—the form $dd^c u$ is reducible. However it is easy to construct an example of a function u such that $dd^c u$ is reducible and neither $\pm \operatorname{Re} u$ nor $\pm \operatorname{Im} u$ is plurisubharmonic.

EXAMPLE 4. Define

$$u(z_1, z_2, z_3) = z_1\bar{z}_2 + z_1\bar{z}_3 + (1+i)(z_2\bar{z}_1 + z_3\bar{z}_1)$$

for $(z_1, z_2, z_3) \in \mathbf{C}^3$. Then

$$dd^c u = 2i(dz_1 \wedge d\bar{z}_2 + dz_1 \wedge d\bar{z}_3 + (1+i)dz_2 \wedge d\bar{z}_1 + (1+i)\, dz_3 \wedge d\bar{z}_1)$$

and hence $dd^c u$ is reducible. Furthermore $(dd^c u)^3 = 0$ and $(dd^c u)^2 \neq 0$. The functions $\operatorname{Re} u = 2\operatorname{Re}(z_1\bar{z}_2 + z_1\bar{z}_3) - \operatorname{Im}(z_1\bar{z}_2 + z_1\bar{z}_3)$ and $\operatorname{Im} u = \operatorname{Re}(z_2\bar{z}_1 + z_3\bar{z}_1)$ are harmonic in $\mathbf{C}^3$ but $\operatorname{Re} u(\lambda, \pm\lambda, 0) = \pm 2|\lambda|^2$ and $\operatorname{Im} u(\lambda, \pm\lambda, 0) = \pm|\lambda|^2$ for $\lambda \in \mathbf{C}$.

4. Proofs of the theorems. First we shall prove Theorem 1. By virtue of Lemmas 2 and 3, $\Delta(z) = \operatorname{Ker} A(z) \cap \operatorname{Ker} B(z)$ for all $z \in \Omega$. Fix $z \in \Omega$. To simplify the writing we omit z in $A(z)$, $B(z)$, α_z and β_z. Since α and β are reducible

$$\operatorname{Ker} A \cap \operatorname{Ker} B = (\operatorname{Ran} A + \operatorname{Ran} B)^{\perp}.$$

The complex rank of α is p. In view of Lemma 1 we may make a unitary change of coordinates such that

$$\alpha = \frac{i}{2} \sum_{j,k=1}^{p} \alpha_{jk}\, dz_j \wedge d\bar{z}_k.$$

Because of Lemma 4 and the fact that $\alpha^p \wedge \beta^q \neq 0$ and $\alpha^p \wedge \beta^{q+1} = 0$, the rank of the matrix $[\beta_{jk}]_{j,k>p}$ is q. Thus the rank of the $(n \times 2n)$ matrix (A, B) is $p + q$ which means that Ran A + Ran B is a $(p + q)$-dimensional complex subspace of $\mathbf{C}^n$. The proof of the first conclusion of Theorem 1 is complete, because z was arbitrary.

Now assume $d\alpha$ and $d\beta$ vanish on $\Delta \times \Delta \times T\Omega$. Let X, Y and Z be vector fields on Ω such that X and Y belong to Δ. Then

$$\begin{aligned} 0 = d\alpha(X, Y, Z) &= X\alpha(Y, Z) - Y\alpha(X, Z) + Z\alpha(X,Y) \\ &\quad - \alpha([X, Y], Z) - \alpha(Y, [X, Z]) + \alpha(X, [Y, Z]), \end{aligned}$$

and the same holds for β. Therefore

$$\alpha([X, Y], Z) = \beta([X, Y], Z) = 0.$$

Consequently $[X, Y]$ belongs to Δ and Δ is involutive. By the Frobenius Integrability Theorem Δ is integrable i.e. there is a foliation $\mathscr{F} = \mathscr{F}(\alpha, \beta)$ of Ω by $\mathscr{C}^1$-submanifolds. If $M \in \mathscr{F}$ and $a \in M$ then $T_a M = \Delta(a)$ is a complex subspace of $\mathbf{C}^n$ by the first part of the theorem. In view of the classical criterion of Levi-Civita this shows that M is a complex submanifold of Ω.

To prove the second theorem construct $\mathscr{F}(\alpha, \beta)$ for $\alpha = dd^c u$ an $\beta = dd^c v$. Let M be a leaf of $\mathscr{F}(\alpha, \beta)$. Then M is a complex submanifold of Ω and hence the operators ∂_M and $\bar{\partial}_M$ are intrinsically defined on M. Moreover if $a \in M$, $T_a M = N_a(dd^c u) \cap N_a(dd^c v)$. If $I\colon M \to \Omega$ denotes inclusion then

$$\partial_M \bar{\partial}_M(u|M) = \partial_M \bar{\partial}_M(I^* u) = I^* \partial \bar{\partial} u = 0.$$

Similarly $\partial_M \bar{\partial}_M(v|M) = 0$. Therefore both $u|M$ and $v|M$ are pluriharmonic on M.

Fix $a \in M$. Let (ϕ, U) be a coordinate system on M such that $a \in U$ and $\phi(a) = 0$. Set $\psi = (\psi_1, \ldots, \psi_n) = \phi^{-1}$. Then

$$\begin{aligned} T_a M &= \partial_0 \psi(\mathbf{C}^{n-p-q}) \subset \operatorname{Ker}\left[\partial^2 u(a)/\partial z_j \partial \bar{z}_k\right]^{\mathrm{tr}} \\ &= \operatorname{Ker} \overline{\left[\partial^2 u(a)/\partial z_j \partial \bar{z}_k)\right]}. \end{aligned}$$

Therefore

$$\frac{\partial}{\partial \bar{\phi}_i}\left(\frac{\partial u}{\partial z_j}\right)(a) = \sum_{k=1}^{n} \frac{\partial^2 u(a)}{\partial z_j \partial \bar{z}_k} \overline{\frac{\partial \psi_k}{\partial w_i}}(0) = 0$$

for $i \in \{1, \ldots, n - p - q\}$ and $j \in \{1, \ldots, n\}$. Thus the restrictions of $\partial u/\partial z_1, \ldots, \partial u/\partial z_n$ to M are holomorphic on M. The same holds when u is replaced by v.

Now we shall prove Theorem 3. Set

$$\alpha = \frac{i}{2}\partial\bar{u} \wedge \bar{\partial}u = \frac{i}{2}\sum \overline{\frac{\partial u}{\partial \bar{z}_j}}\,\frac{\partial u}{\partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

and

$$\beta = \frac{i}{2}\partial\,\bar{\partial}u = \frac{1}{4}dd^c u.$$

Then

$$A_z = \left[\frac{\partial u}{\partial \bar{z}_j}(z) \cdot \overline{\frac{\partial u}{\partial \bar{z}_k}(z)}\right]$$

is the matrix of the mapping, $\mathbf{C}^n \ni w \to \langle w, a_z\rangle a_z \in \mathbf{C}^n$ where

$$a_z = \left(\frac{\partial u}{\partial \bar{z}_1}(z), \ldots, \frac{\partial u}{\partial \bar{z}_n}(z)\right).$$

Hence the rank of A is one and A is Hermitian. Furthermore

$$d\alpha = -\tfrac{1}{2}\mathrm{Re}(\partial\bar{u} \wedge dd^c u)$$

so we can apply Theorem 1 to get a foliation $\mathscr{F}$ of Ω of complex codimension $p + 1$.

Let M be a leaf of M and let I: $M \to \Omega$ denotes inclusion. Then $\bar{\partial}_M u = 0$ because $0 = I^*(\partial\bar{u} \wedge \bar{\partial}u) = \overline{(\bar{\partial}_M u)} \wedge \bar{\partial}_M u$.

The following example shows that the rank of the Hessian matrix of the function v in Theorem 2 may vary.

EXAMPLE 5. Define $v(z_1, z_2, z_3) = |z_1 + z_3|^4 + |z_2|^4$. The function v is plurisubharmonic and $(dd^c v)^3 = 0$ in $\mathbf{C}^3$. The rank of the form $dd^c v$ is 2 on the set $\{(z_1, z_2, z_3)\colon z_2(z_1 + z_3) \neq 0\}$. The rank drops to 1 when $z_2(z_1 + z_3) = 0$ and $z_2 \neq z_1 + z_3$ and to 0 when $z_2 = z_1 + z_3 = 0$. Put $u(z_1, z_2, z_3) = |z_1 + z_3|^2 + |z_2|^2$. The functions u and v satisfy the assumptions of Theorem 2 with $p = 2$ and $q = 0$. Thus there is a complex one-dimensional foliation of $\mathbf{C}^3$ along the leaves of which v is harmonic and the derivatives $\partial v/\partial z_j$ are holomorphic. One can say that $dd^c u$ which has a constant rank removes singularities coming from a drop in the rank of the Hessian of v.

REFERENCES

1. E. Bedford and M. Kalka, *Foliations and complex Monge-Ampère equations*, Comm. Pure Appl. Math. **30** (1977), 543–571.

2. E. Bedford and D. Burns, *Holomorphic mappings of annuli in* $\mathbf{C}^n$ *and the associated extremal function*, Ann. Scuola Norm. Sup. Pisa (4) **6** (1979), 381–414.

3. J. Kalina, J. Ławrynowicz, E. Ligocka and M. Skwarczyński, *On some biholomorphic invariants in the analysis on manifolds*, Lecture Notes in Math., vol. 789, Springer, Berlin and New York, 1980, pp. 224–249.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DUBLIN, TRINITY COLLEGE, DUBLIN 2, IRELAND

Current address: Department of Mathematics, University College, Belfield, Dublin 4, Ireland