COMPLEX FOLIATIONS GENERATED BY (1, 1)-FORMS

M. KLIMEK

Abstract. Complex foliations generated by (1, 1)-forms are studied in order to describe geometric properties of complex partial differential equations of Monge-Ampère type.

1. Introduction. Let \( \alpha = \frac{i}{2} \sum \alpha_{jk} \, dz_j \wedge d\bar{z}_k \) be a \( C^1 \)-differential form on an open set \( \Omega \subset \mathbb{C}^n \). By \( \alpha^m \) we shall denote the \( m \)th exterior power of \( \alpha \) (\( \alpha^0 \equiv 1 \)). If \( z \in \Omega \) then \( N_z(\alpha) \) is the nullity space of \( \alpha \) at \( z \) i.e.

\[
N_z(\alpha) = \{ x \in \mathbb{C}^n : \alpha_z(x, y) = 0 \text{ for all } y \in \mathbb{C}^n \}.
\]

If for each \( z \in \Omega \), the linear operator corresponding to the matrix \( [\alpha_{jk}(z)] \) is reduced by its range we say that \( \alpha \) is reducible. A detailed geometric characterization of the notion is given in the third section. In particular it is proved there that if the complex rank of \( \alpha \) is \( p \) then the mapping \( z \to N_z(\alpha) \) is a \( \mathbb{C}^1 \)-distribution of real dimension \( 2(n - p) \) if and only if \( \alpha \) is reducible.

The aim of this paper is to study complex foliations (i.e. \( \mathbb{C}^1 \)-foliations whose leaves are complex manifolds) generated by reducible differential forms. We are particularly interested in forms \( dd^c u \) where \( u \) is a complex valued \( C^3 \)-function satisfying the Monge-Ampère equation

\[
(1) \quad (dd^c u)^{p+1} = 0
\]

in an open set \( \Omega \subset \mathbb{C}^n \) with \( 0 < p < n \). (Recall that \( d = \partial + \bar{\partial} \) and \( d^c = i(\bar{\partial} - \partial) \).)

It is known (see [1]) that if the above equation is nondegenerate i.e.

\[
(2) \quad (dd^c u)^p \neq 0
\]

in \( \Omega \) and \( \text{Im } u \) is plurisubharmonic then there exists a complex foliation \( \mathcal{F} \) of \( \Omega \) such that \( u \) is pluriharmonic along the leaves of \( \mathcal{F} \). We shall prove that this remains true in a more general case when \( dd^c u \) is reducible. Furthermore if for a complex function \( u \) of class \( C^3 \) the nullity spaces of \( dd^c u \) generate a complex foliation of codimension \( p \) then \( dd^c u \) must be reducible.

We shall examine the system of equations of Monge-Ampère type

\[
(3) \quad (dd^c u)^p \wedge (dd^c v)^{q+1} = 0.
\]

\[
(4) \quad (dd^c u)^{p+1} = 0
\]
with the additional nondegeneracy condition
\[(dd^c u)^p \wedge (dd^c v)^q \neq 0\]
where \(p\) and \(q\) are nonnegative integers, \(0 < p + q < n\), \(u\) and \(v\) are complex functions on an open subset \(\Omega\) of \(\mathbb{C}^n\). The conditions (4) and (5) imply that the rank of \(dd^c u\) is constant. Nevertheless the rank of \(dd^c v\) may vary. In particular the equation \((dd^c v)^{p+q+1} = 0\) may be degenerate (cf. Example 5). In general, very little is known about such equations (see [1, 2]). We shall prove that if \(dd^c u\) and \(dd^c v\) are reducible \(\mathcal{E}^1\)-forms satisfying (3), (4) and (5) then there is a complex foliation \(\mathcal{F}\) of \(\Omega\) such that both \(u\) and \(v\) are pluriharmonic when restricted to any leaf of \(\mathcal{F}\).

2. Main results. We shall prove the following.

**Theorem 1.** Let \(\alpha = (i/2)\sum \alpha_{jk} \ dz_j \wedge d\bar{z}_k\) and \(\beta = (i/2)\sum \beta_{jk} \ dz_j \wedge d\bar{z}_k\) be reducible \(\mathcal{E}^1\)-forms on an open subset \(\Omega\) of \(\mathbb{C}^n\). Let \(A = [\alpha_{jk}]^u\) and \(B = [\beta_{jk}]^v\). Assume \(p\) and \(q\) are nonnegative integers such that \(0 < p + q < n\) and the conditions
\[\alpha^p \wedge \beta^q \neq 0, \quad \alpha^p \wedge \beta^q+1 = 0, \quad \alpha^{p+1} = 0\]
are satisfied in \(\Omega\). Then \(\Delta (z) = N_c(\alpha) \cap N_c(\beta)\) is a \(\mathcal{E}^1\)-distribution over \(\Omega\) of (real) dimension \(2(n - p - q)\) and \(\Delta (z) = \text{Ker} A(z) \cap \text{Ker} B(z)\) for all \(z \in \Omega\). If \(d\alpha\) and \(d\beta\) vanish on \(\Delta(z) \times \Delta(z) \times \mathbb{C}^n\) for each \(z \in \Omega\) the distribution \(\Delta\) is integrable and generates a complex foliation \(\mathcal{F}\) of \(\Omega\) of complex codimension \(p + q\).

The above theorem generalizes some earlier results obtained by Bedford, Kalka [1] and Kalina [3]. Under the additional assumption that \(\alpha = \beta\) and \(\text{Im} \ \alpha > 0\) it has been proved in [1]. For real forms \(\alpha\) of class \(\mathcal{E}^3\) and \(\beta = dd^c u\) with \(\text{Im} u\) plurisubharmonic the result has been shown by Kalina [3]. However the way in which we prove Theorem 1 in this paper is different from the methods used in [1 and 3].

Theorem 1 yields the following two results.

**Theorem 2.** Let \(u\) and \(v\) be complex functions of class \(\mathcal{E}^3\) on \(\Omega \subset \mathbb{C}^n\) such that \(dd^c u\) and \(dd^c v\) are reducible. Let \(p\) and \(q\) be nonnegative integers such that \(0 < p + q < n\) and the conditions (3), (4) and (5) are satisfied in \(\Omega\). Then there exists a complex foliation \(\mathcal{F}\) of \(\Omega\) by complex submanifolds of codimension \(p + q\) such that for any leaf \(M\) of \(\mathcal{F}\), the restrictions of \(u\) and \(v\) to \(M\) are pluriharmonic on \(M\) and \(\partial \text{Re} u/\partial z_j, \partial \text{Re} v/\partial z_j, \partial \text{Im} u/\partial z_j, \partial \text{Im} v/\partial z_j\) are holomorphic on \(M\) for \(j = 1, 2, \ldots, n\).

**Theorem 3.** Suppose \(u\) is a complex \(\mathcal{E}^3\)-function on \(\Omega\) such that \(\partial u\) is reducible and for some \(p\) (\(0 < p < n\))
\[\partial u \wedge \partial u \wedge (\partial u)^p = 0, \quad \partial u \wedge \partial u \wedge (\partial u)^{p+1} = 0\]
in \(\Omega\). Then there is a complex foliation \(\mathcal{F}\) on \(\Omega\) of (complex) codimension \(p + 1\) such that for every leaf \(M\) of \(\mathcal{F}\), \(u|\ M\) is holomorphic on \(M\).

If \(u = v\) and \(\text{Im} u\) is plurisubharmonic Theorems 2 and 3 reduce to Theorems 2.4 and 5.1 in [1] respectively.

3. Reducible forms. For any \(\mathbb{C}\)-linear operator \(A: \mathbb{C}^n \to \mathbb{C}^n\), \(\text{Ker} A\) and \(\text{Ran} A\) denote the kernel and range of \(A\) respectively. We consider \(\mathbb{C}^n\) as a Hilbert space
with the inner product $\langle z, w \rangle = \sum z_j \overline{w}_j$. Every operator $A$ induces the orthogonal decompositions $C^n = \text{Ran } A \oplus \text{Ker } A^* = \text{Ran } A^* \oplus \text{Ker } A$. Moreover if $U$ is a unitary operator and $P$ is a projection then $UPU^*$ is again a projection. Hence we have

**Lemma 1.** If $A: C^n \to C^n$ is a $C$-linear operator of rank $p < n$ then the following conditions are equivalent:

1. $\text{Ran } A$ reduces $A$,
2. $\text{Ran } A = \text{Ran } A^*$ and $\text{Ker } A = \text{Ker } A^*$,
3. $A$ commutes with the orthogonal projection onto $\text{Ran } A$,
4. there is a unitary operator $U: C^n \to C^n$ and a $C$-linear isomorphism $B: C^p \to C^p$ such that for all $(z, w) \in C^p \times C^{n-p}$

$$B(z, 0) = UAU^*(z, w).$$

If $A$ satisfies any of the above conditions, we will say it is reducible.

**Example 1.** If $A$ is reducible then $A''$ is also reducible.

**Example 2.** If $A = S + iT$ where both $S$ and $T$ are Hermitian operators and either $S$ or $T$ is semidefinite then $A$ is reducible. To see this observe if $z \in \text{Ker } A$ then $\langle Az, z \rangle = 0$ and hence $\langle Sz, z \rangle = \langle Tz, z \rangle = 0$ since $S$ and $T$ are Hermitian. If—for instance—$S$ is positive semidefinite, it has a square root $S^{1/2}$, so that $\langle S^{1/2}z, S^{1/2}z \rangle = 0$ and hence $z \in \text{Ker } S$. Consequently $z \in \text{Ker } T$. Therefore $z \in \text{Ker } A^*$.

**Example 3.** Suppose $A$ is normal i.e. it commutes with its conjugate. Since eigenvectors of $A$ belonging to different eigenvalues are orthogonal, $A$ can be diagonalized by a unitary transformation and so it is reducible.

**Lemma 2.** If $A: C^n \to C^n$ is $C$-linear then $N = \{z \in C^n: \langle Az, w \rangle = \langle Aw, z \rangle \text{ for all } w \in C^n\} = \text{Ker } A \cap \text{Ker } A^*$.

**Proof.** Set $S = \frac{1}{2}(A + A^*)$ and $T = \frac{(A - A^*)}{2i}$. Then both $S$ and $T$ are selfadjoint and $A = S + iT$. Clearly $\text{Ker } A \cap \text{Ker } A^* \subset N$. To prove the opposite inclusion it is enough to show that $N \subset \text{Ker } S \cap \text{Ker } T$. Fix $z \in N$. Then for all $w \in C^n$

$$\langle Sz, w \rangle - \langle Sw, z \rangle + i(\langle Tz, w \rangle - \langle Tw, z \rangle) = 0.$$

But $\langle Sz, w \rangle - \langle Sw, z \rangle = 2i \text{Im}(\langle Sz, w \rangle)$ and $\langle Tz, w \rangle - \langle Tw, z \rangle = 2i \text{Im}(\langle Tz, w \rangle)$. Moreover $\langle x, y \rangle = \text{Im}(x, -iy) + i \text{Im}(x, y)$ for all $x, y \in C^n$. Thus for all $w \in C^n$,

$$\langle Sz, w \rangle = \langle Tz, w \rangle = 0$$

and we are done.

**Corollary 1.** $N = (\text{Ran } A + \text{Ran } A^*)^\perp$.

**Lemma 3.** If $\alpha = (i/2)\sum \alpha_{jk} dz_j \wedge d\overline{z}_k$ is a $(1, 1)$-form with constant coefficients and $A = [\alpha_{jk}]^\nu$ then

$$\alpha(x, y) = i \left( \frac{1}{2} (\langle Ax, y \rangle - \langle Ay, x \rangle) \right).$$

**Proof.** If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then $dz_j \wedge d\overline{z}_k(x, y) = x_j \overline{y}_k - y_j \overline{x}_k$ which yields the lemma.
For a positive integer \(p\) define \(c_p\) by setting \(c_p = 2^{-p}\) if \(p\) is even and \(c_p = i2^{-p}\) if \(p\) is odd. Then for multi-indices \(J = (j_1, \ldots, j_p)\) and \(K = (k_1, \ldots, k_p)\) we have
\[
\left(\frac{i}{2}\right)^p \prod_{j=1}^p dz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_p} = c_p dz^J \wedge d\bar{z}^K.
\]
As a consequence of this equality and the definition of the determinant function we get

**Lemma 4.** If \(\alpha = (i/2)\sum_{j,k} \alpha_{jk} dz_j \wedge d\bar{z}_k\) has constant coefficients then
\[
\alpha^p = p!c_p \sum_{J,K} \alpha_{J,K} dz^J \wedge d\bar{z}^K
\]
where the summation is taken over all multi-indices \(J = (j_1, \ldots, j_p)\) and \(K = (k_1, \ldots, k_p)\) such that \(1 \leq j_1 < \cdots < j_p \leq n\) and \(1 \leq k_1 < \cdots < k_p \leq n\) and \(\alpha_{J,K} = \det[\alpha_{j,k}]_{j \in J, k \in K}\).

Let \(\Omega\) be an open subset of \(\mathbb{C}^n\) and let \(\alpha = (i/2)\sum_{j,k} \alpha_{jk} dz_j \wedge d\bar{z}_k\) be a \(\mathcal{C}^1\)-form which is reducible (i.e. \([\alpha_{jk}(z)]\) is reducible for all \(z \in \Omega\)). Then \(A(z) = [\alpha_{jk}(z)]^T\) is reducible for each \(z \in \Omega\) and the above lemmas imply

**Corollary 2.** If \(\alpha\) is a reducible \(\mathcal{C}^1\)-form on \(\Omega\) and \(z \in \Omega\) then the nullity space of \(\alpha\) at \(z\) is a complex subspace of \(\mathbb{C}^n\). Moreover
\[
N_\Omega(\alpha) = (\text{Ran} A(z))^\perp = \text{Ker} A(z).
\]
As a consequence of Corollaries 1 and 2 and Lemma 4 we obtain

**Corollary 3.** Let \(\alpha\) be a \((1, 1)\)-form of class \(\mathcal{C}^1\) on an open set \(\Omega \subset \mathbb{C}^n\) such that the complex rank of \(\alpha\) is \(p\) (i.e. \(\alpha^p \neq 0, \alpha^{p+1} = 0\)). Then the mapping \(z \mapsto N_\Omega(\alpha)\) is a \(\mathcal{C}^1\)-distribution of real dimension \(2(n - p)\) if and only if \(\alpha\) is reducible.

Since \(d = \partial + \bar{\partial}\) and \(d^c = i(\bar{\partial} - \partial)\), \(dd^c\) is reducible. Let \(u\) be a complex \(\mathcal{C}^3\)-function on \(\Omega\) such that at least one of the functions \(\text{Re } u\), \(-\text{Re } u\), \(\text{Im } u\), \(-\text{Im } u\) is plurisubharmonic. Then—in view of Example 2—the form \(dd^c u\) is reducible. However it is easy to construct an example of a function \(u\) such that \(dd^c u\) is reducible and neither \(\pm \text{Re } u\) nor \(\pm \text{Im } u\) is plurisubharmonic.

**Example 4.** Define
\[
u(z_1, z_2, z_3) = z_1\bar{z}_2 + z_1\bar{z}_3 + (1 + i)(z_2\bar{z}_1 + z_3\bar{z}_1)
\]
for \((z_1, z_2, z_3) \in \mathbb{C}^3\). Then
\[
 dd^c u = 2i(dz_1 \wedge d\bar{z}_2 + dz_1 \wedge d\bar{z}_3 + (1 + i)dz_2 \wedge d\bar{z}_1 + (1 + i)dz_3 \wedge d\bar{z}_1)\]
and hence \(dd^c u\) is reducible. Furthermore \((dd^c u)^3 = 0\) and \((dd^c u)^2 \neq 0\). The functions \(\text{Re } u = 2\text{Re}(z_1\bar{z}_2 + z_1\bar{z}_3) - \text{Im}(z_1\bar{z}_2 + z_1\bar{z}_3)\) and \(\text{Im } u = \text{Re}(z_2\bar{z}_1 + z_3\bar{z}_1)\) are harmonic in \(\mathbb{C}^3\) but \(\text{Re } u(\lambda, \pm 0) = \pm 2|\lambda|^2\) and \(\text{Im } u(\lambda, \pm 0) = \pm |\lambda|^2\) for \(\lambda \in \mathbb{C}\).

**4. Proofs of the theorems.** First we shall prove Theorem 1. By virtue of Lemmas 2 and 3, \(\Delta(z) = \text{Ker} A(z) \cap \text{Ker} B(z)\) for all \(z \in \Omega\). Fix \(z \in \Omega\). To simplify the writing we omit \(z\) in \(A(z), B(z), \alpha, \beta\). Since \(\alpha\) and \(\beta\) are reducible
\[
\text{Ker } A \cap \text{Ker } B = (\text{Ran } A + \text{Ran } B)^\perp.
\]
The complex rank of $\alpha$ is $p$. In view of Lemma 1 we may make a unitary change of coordinates such that

$$\alpha = \frac{i}{2} \sum_{j,k=1}^{p} \alpha_{j,k} \, dz_j \wedge dz_k.$$ 

Because of Lemma 4 and the fact that $\alpha^p \wedge \beta^q \neq 0$ and $\alpha^p \wedge \beta^{q+1} = 0$, the rank of the matrix $[\beta_{jk}]_{j,k=p}$ is $q$. Thus the rank of the $(n \times 2n)$ matrix $(A, B)$ is $p + q$ which means that $\text{Ran} \, A + \text{Ran} \, B$ is a $(p + q)$-dimensional complex subspace of $\mathbb{C}^n$. The proof of the first conclusion of Theorem 1 is complete, because $z$ was arbitrary.

Now assume $d\alpha$ and $d\beta$ vanish on $\Delta \times \Delta \times T\Omega$. Let $X, Y$ and $Z$ be vector fields on $\Omega$ such that $X$ and $Y$ belong to $\Delta$. Then

$$0 = d\alpha(X, Y, Z) = X\alpha(Y, Z) - Y\alpha(X, Z) + Z\alpha(X, Y)$$

$$-\alpha([X, Y], Z) - \alpha(Y, [X, Z]) + \alpha(X, [Y, Z]),$$

and the same holds for $\beta$. Therefore

$$\alpha([X, Y], Z) = \beta([X, Y], Z) = 0.$$ 

Consequently $[X, Y]$ belongs to $\Delta$ and $\Delta$ is involutive. By the Frobenius Integrability Theorem $\Delta$ is integrable i.e. there is a foliation $\mathcal{F} = \mathcal{F}(\alpha, \beta)$ of $\Omega$ by $\mathbb{C}^1$-submanifolds. If $M \in \mathcal{F}$ and $a \in M$ then $T_aM = \Delta(a)$ is a complex subspace of $\mathbb{C}^n$ by the first part of the theorem. In view of the classical criterion of Levi-Civita this shows that $M$ is a complex submanifold of $\Omega$.

To prove the second theorem construct $\mathcal{F}(\alpha, \beta)$ for $\alpha = dd^c u$ an $\beta = dd^c v$. Let $M$ be a leaf of $\mathcal{F}(\alpha, \beta)$. Then $M$ is a complex submanifold of $\Omega$ and hence the operators $\partial_M$ and $\overline{\partial}_M$ are intrinsically defined on $M$. Moreover if $a \in M$, $T_aM = N_a(dd^c u) \cap N_a(dd^c v)$. If $I: M \to \Omega$ denotes inclusion then

$$\partial_M \overline{\partial}_M (u|M) = \partial_M \overline{\partial}_M (I^*u) = I^* \partial \overline{\partial} u = 0.$$ 

Similarly $\partial_M \overline{\partial}_M (v|M) = 0$. Therefore both $u|M$ and $v|M$ are pluriharmonic on $M$.

Fix $a \in M$. Let $(\phi, U)$ be a coordinate system on $M$ such that $a \in U$ and $\phi(a) = 0$. Set $\psi = (\psi_1, \ldots, \psi_n) = \phi^{-1}$. Then

$$T_aM = \partial_0 \psi(\mathbb{C}^{n-p-q}) \subset \text{Ker} \left[ \partial^2 u(a)/\partial z_j \partial \bar{z}_k \right]|_\mathbb{C}^{n-p-q}$$

$$= \text{Ker} \left[ \partial^2 u(a)/\partial z_j \partial \bar{z}_k \right].$$

Therefore

$$\frac{\partial}{\partial \phi_i} \left( \frac{\partial u}{\partial z_j} \right)(a) = \sum_{k=1}^{n} \frac{\partial^2 u(a)}{\partial z_j \partial \bar{z}_k} \frac{\partial \psi_k}{\partial w_i}(0) = 0$$

for $i \in \{1, \ldots, n - p - q \}$ and $j \in \{1, \ldots, n \}$. Thus the restrictions of $\partial u/\partial z_1, \ldots, \partial u/\partial z_n$ to $M$ are holomorphic on $M$. The same holds when $u$ is replaced by $v$. 

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Now we shall prove Theorem 3. Set
\[
\alpha = \frac{i}{2} \partial \overline{\partial} u = \frac{i}{2} \sum \frac{\partial u}{\partial z_j} \frac{\partial u}{\partial \overline{z}_k} \, dz_j \wedge d\overline{z}_k
\]
and
\[
\beta = \frac{i}{2} \partial \overline{\partial} u = \frac{1}{4} dd^c u.
\]
Then
\[
A_z = \left[ \frac{\partial u}{\partial z_j}(z), \frac{\partial u}{\partial \overline{z}_k}(z) \right]
\]
is the matrix of the mapping, \( C^n \ni w \to (w, a_z) \in C^n \) where
\[
a_z = \left( \frac{\partial u}{\partial z_1}(z), \ldots, \frac{\partial u}{\partial z_n}(z) \right).
\]
Hence the rank of \( A \) is one and \( A \) is Hermitian. Furthermore
\[
d\alpha = -\frac{1}{2} \text{Re}(\partial \overline{u} \wedge dd^c u)
\]
so we can apply Theorem 1 to get a foliation \( \mathcal{F} \) of \( \Omega \) of complex codimension \( p + 1 \).

Let \( M \) be a leaf of \( M \) and let \( I : M \to \Omega \) denotes inclusion. Then \( \partial_M u = 0 \) because
\[
0 = I^*(\partial \overline{u} \wedge \partial u) = (\overline{\partial_M u}) \wedge \overline{\partial_M u}.
\]
The following example shows that the rank of the Hessian matrix of the function \( v \) in Theorem 2 may vary.

**Example 5.** Define \( v(z_1, z_2, z_3) = |z_1 + z_3|^4 + |z_2|^4 \). The function \( v \) is plurisubharmonic and \((dd^c v)^3 = 0 \) in \( C^3 \). The rank of the form \( dd^c v \) is 2 on the set \( \{(z_1, z_2, z_3) : z_2(z_1 + z_3) \neq 0\} \). The rank drops to 1 when \( z_2(z_1 + z_3) = 0 \) and \( z_2 \neq z_1 + z_3 \) and to 0 when \( z_2 = z_1 + z_3 = 0 \). Put \( u(z_1, z_2, z_3) = |z_1 + z_3|^2 + |z_2|^2 \). The functions \( u \) and \( v \) satisfy the assumptions of Theorem 2 with \( p = 2 \) and \( q = 0 \). Thus there is a complex one-dimensional foliation of \( C^3 \) along the leaves of which \( v \) is harmonic and the derivatives \( \partial v/\partial z_j \) are holomorphic. One can say that \( dd^c u \) which has a constant rank removes singularities coming from a drop in the rank of the Hessian of \( v \).

**References**


Department of Mathematics, University of Dublin, Trinity College, Dublin 2, Ireland

Current address: Department of Mathematics, University College, Belfield, Dublin 4, Ireland