COMPACT 3-MANIFOLDS WITH INFINITELY-GENERATED GROUPS OF SELF-HOMOTOPY-EQUIVALENCES

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ABSTRACT. Examples are constructed of compact 3-manifolds with boundary whose groups of self-homotopy-equivalences are not finitely-generated.

For a finite CW-complex $X$, let $\mathcal{E}(X)$ denote the space of basepoint-preserving homotopy equivalences from $X$ to $X$, and let $\mathcal{G}(X)$ denote the group of homotopy equivalences $\pi_0(\mathcal{E}(X))$. An obvious question is: Under what conditions on $X$ must $\mathcal{G}(X)$ be finitely-generated? Sullivan [7] and Wilkerson [10] showed that if $X$ is simply-connected, then $\mathcal{G}(X)$ is finitely-presented. For non-simply-connected complexes, however, $\mathcal{G}(X)$ can be infinitely-generated (have no finite generating set) even for seemingly uncomplicated examples. Frank and Kahn [3] showed that $\mathcal{G}(S^1 \vee S^p \vee S^{2p-1})$ is infinitely-generated when $p \geq 2$, and in [5] the author gave infinitely many examples of finite four-dimensional $K(\pi, 1)$-complexes with Aut$(\pi)$ and, hence, $\mathcal{G}(K(\pi, 1))$ infinitely-generated. In [5], it was asked whether there was an example of an aspherical 2-complex $X$ with $\mathcal{G}(X)$ infinitely-generated. Recently, such examples were found by Brunner and Ratcliffe [2].

These various examples show two distinct ways that $\mathcal{G}(X)$ can fail to be finitely-generated. In the Frank and Kahn examples, $\pi_{2p-1}(S^1 \vee S^p \vee S^{2p-1})$ is quite large—it is infinitely-generated as a $\mathbb{Z}\pi_1(S^1 \vee S^p \vee S^{2p-1})$-module—and many elements of $\mathcal{G}(S^1 \vee S^p \vee S^{2p-1})$ arise by mapping the $S^{2p-1}$ using an element of $\pi_{2p-1}(S^1 \vee S^p \vee S^{2p-1})$. In the aspherical examples, Aut$(\pi_1(X))$ is infinitely-generated and the asphericity forces $\mathcal{G}(X) \to \text{Aut}(\pi_1(X))$ to be surjective. Clearly, if $\mathcal{G}(X) \to \text{Aut}(\pi_1(X))$ is surjective then so is

$$\mathcal{G}(X \vee S^n) \to \text{Aut}(\pi_1(X \vee S^n))$$

for $n \geq 2$,

so one can produce nonaspherical examples in dimension two. What is not apparent from these examples is the answer to the following question posed in [2].

**Question.** Is there a finite two-dimensional complex $X$ with Aut$(\pi_1(X))$ finitely-generated but $\mathcal{G}(X)$ not finitely-generated?
For a 3-manifold $M$ let $M'$ denote the result of removing from $M$ the interiors of two disjoint closed 3-balls tamely-imbedded in the interior of $M$. Our main result is

**Theorem 1.** Let $M$ be a compact aspherical 3-manifold-with-boundary, such that $\text{Out}(\pi_1(M))$ is finite and $\pi_1(M)$ admits a surjective homomorphism onto $\mathbb{Z} \times \mathbb{Z}$. Then $\mathcal{G}(M')$ is infinitely-generated.

Many instances of Theorem 1 are given in

**Corollary.** Let $M$ be a compact orientable 3-manifold-with-boundary such that the interior of $M$ admits a complete hyperbolic structure with finite volume, and such that $\pi_1(M)$ admits a surjective homomorphism onto $\mathbb{Z} \times \mathbb{Z}$. Then $\mathcal{G}(M')$ is infinitely-generated.

**Proof.** $M$ is aspherical since its interior is, and $\text{Out}(\pi_1(M))$ is well known to be finite [6, p. 116; 8, p. 5.31]. □

For example, the (compact) complement of the Whitehead link and the (compact) complement of the Borromean rings are familiar 3-manifolds which satisfy the hypotheses of the theorem and the corollary.

Since any compact 3-manifold-with-boundary has the homotopy type of a finite 2-complex, and $\text{Out}(\pi_1(M'))$ finitely-generated implies $\text{Aut}(\pi_1(M'))$ finitely-generated, Theorem 1 answers the question of Brunner and Ratcliffe in the affirmative.

We will give the proof of Theorem 1 in §1, making use of two auxiliary theorems. These theorems, which are of independent interest, are proved in §§2 and 3. I wish to thank Andy Miller for helpful discussions concerning Theorem 2(b).

**1. Proof of Theorem 1.** Write $\pi$ for $\pi_1(M, *) \cong \pi_1(M', *)$. Let $\Phi: \mathcal{G}(M') \rightarrow \text{Aut}(\pi)$ be the homomorphism defined by $\Phi(f) = f_\pi$. Let $\mathcal{G}_1(M') = \Phi^{-1}(\{1\})$ and $\mathcal{G}_{\text{Inn}}(M') = \Phi^{-1}(\text{Inn}(\pi))$. Since $\text{Out}(\pi)$ is finite, $\mathcal{G}_{\text{Inn}}(M')$ has finite index in $\mathcal{G}(M')$, so to prove the theorem it suffices to show $\mathcal{G}_{\text{Inn}}(M')$ is infinitely-generated.

Let $M_1 = M$ if $M$ is orientable, otherwise let $M_1$ be the orientable double cover of $M$. Now $M_1$ is compact, orientable, and has a boundary component which is not a 2-sphere. Therefore, $H_1(M_1; \mathbb{Z})$ is infinite so $M_1$ is sufficiently large. Therefore, the center of $\pi_1(M_1)$ is finitely-generated [9]. This implies that the center of $\pi$ is finitely-generated. Using Theorem 2(b), which will be stated and proved in §2, we see that $\mathcal{G}_{\text{Inn}}(M')$ is infinitely-generated if $\mathcal{G}_1(M')$ is.

Let $\text{Aut}_\pi(\pi_2(M'))$ be the group of $\pi$-module automorphisms of $\pi_2(M')$. We will prove that the natural homomorphism $\mathcal{G}_1(M') \rightarrow \text{Aut}_\pi(\pi_2(M'))$ is surjective. Let $K$ be a finite 2-complex having the homotopy type of $M$; then $K$ is aspherical and $K' = K \lor S^2 \lor S^2$ has the homotopy type of $M'$. Since $\pi$ has cohomological dimension two, the $k$-invariant $k(K')$ is zero. As shown in [2], this implies $\mathcal{G}_1(K') \rightarrow \text{Aut}_\pi(\pi_2(K'))$ is surjective. (This surjectivity can be proved directly for $K'$ without difficulty: just define a homotopy equivalence that is the identity on $K$ and induces the desired automorphism on $\pi_2(K')$.) Therefore, $\mathcal{G}_1(M') \rightarrow \text{Aut}_\pi(\pi_2(M'))$ is surjective, so $\mathcal{G}_1(M')$ is infinitely-generated if $\text{Aut}_\pi(\pi_2(M'))$ is. But $\pi_2(M') \cong \pi_2(K') \cong \mathbb{Z}\pi \oplus \mathbb{Z}\pi$, so $\text{Aut}_\pi(\pi_2(M')) \cong \text{GL}_2(\mathbb{Z}\pi)$, the group of $2 \times 2$ invertible matrices.
with entries in \( \mathbb{Z} \pi \). We apply Theorem 3, which will be stated and proved in §3, with \( G = \pi \) to show that \( \text{GL}_2(\mathbb{Z} \pi) \) is infinitely-generated. This completes the proof of Theorem 1. \( \square \)

2. Proof of Theorem 2. While part (a) of Theorem 2 is not needed in the proof of Theorem 1, it is of independent interest and can be proved without much more work than that needed for part (b). A special case of part (a) appears in [4].

**Theorem 2.** Let \( L \) be a finite-dimensional locally-finite connected simplicial complex. Then:

(a) If \( \pi_1(L, \ast) \) is centerless, then \( \mathcal{G}_{\text{Inn}}(L) \cong \mathcal{G}_1(L) \times \pi_1(L, \ast) \).

(b) If the center of \( \pi_1(L, \ast) \) is finitely-generated, then \( \mathcal{G}_{\text{Inn}}(L) \) is infinitely-generated if and only if \( \mathcal{G}_1(L) \) is infinitely-generated.

**Proof.** Write \( \pi \) for \( \pi_1(L, \ast) \). Replacing \( L \) by the interior of a regular neighborhood of \( L \), we may assume \( L \) is a triangulated open manifold, and the basepoint \( \ast \) is a vertex of \( L \). Let \( N \) be a regular neighborhood in \( L \) of the 1-skeleton of \( L \). Define \( \alpha: \pi \to \mathcal{G}_{\text{Inn}}(L) \) as follows. For each \( \sigma \in \pi \), choose an isotopy \( H_\sigma: L \times [0,1] \to L \) starting at the identity map \( 1_L \) so that the trace of \( H_\sigma \) (the homotopy class of the restriction of \( H_\sigma \) to \( \ast \times [0,1] \)) equals \( \sigma^{-1} \), and so that the restriction of \( H_\sigma \) to \( (L - \text{int}(N)) \times \{t\} \) equals the identity for all \( t \in [0,1] \). Let \( h_\sigma(x) = H_\sigma(x,1) \). Note that for \( \tau \in \pi \), \( (h_\sigma)^{-1}(\tau) = \sigma \tau^{-1} \), so we can define \( \alpha(\sigma) = (h_\sigma) \).

We will now show that \( \alpha \) is a homomorphism. For homotopies \( G, H: L \times [0,1] \to L \) with \( G(x,0) = H(x,0) \), we define \( (G * H)(x,t) \) to be \( G(x,2t) \) if \( 0 \leq t \leq \frac{1}{2} \) and to be \( H(x,2t-1) \) if \( \frac{1}{2} \leq t \leq 1 \). We define \( G(x,t) \) to be \( G(x,1-t) \). Suppose \( \sigma, \tau \in \pi \).

Then \( H_\sigma * (h_\tau \circ H_\tau) \) is a homotopy from \( 1_L \) to \( h_\sigma h_\tau \) with trace \( \sigma^{-1} \cdot \tau^{-1} \cdot \sigma^{-1} = (\sigma \tau)^{-1} \). Therefore the trace of \( (h_\sigma)^{-1} * (h_\sigma \circ h_\tau) \) is 1 so \( \langle h_\sigma \rangle = \langle h_\sigma h_\tau \rangle = \langle h_\sigma \rangle \langle h_\tau \rangle \).

Next, we will show that if \( \langle f \rangle \in \mathcal{G}_1(L) \) and \( \langle h_\sigma \rangle \in \text{im}(\alpha) \), then \( \langle f \rangle \langle h_\sigma \rangle = \langle h_\sigma \rangle \langle f \rangle \). We may choose \( f \) within its homotopy class so that \( f|_N = 1_N \). A homotopy \( G: f h_\sigma = h_\sigma f \) is defined by \( G(x,t) = h_\sigma(x) \) if \( x \in N \) and \( G(x,t) = h_\sigma(f(x),t) \) if \( x \in L - \text{int}(N) \).

Let \( \mu: \pi \to \text{Inn}(\pi) \) send \( \sigma \) to the inner automorphism \( \mu(\sigma)(\tau) = \sigma \tau \sigma^{-1} \). When \( \pi \) is centerless, \( \mu \) is an isomorphism and \( \alpha \circ \mu^{-1} \) provides a splitting in the exact sequence \( 1 \to \mathcal{G}_1(L) \to \mathcal{G}_{\text{Inn}}(L) \to \text{Inn}(\pi) \to 1 \). Since \( \text{im}(\alpha) \) commutes with \( \mathcal{G}_1(L) \), this establishes part (a). For (b), observe that \( \mathcal{G}_1(L) \cap \text{im}(\alpha) \) is the image under \( \alpha \) of the center of \( \pi \), so it will be finitely-generated when the center of \( \pi \) is. But inclusion induces an isomorphism \( \mathcal{G}_1(L)/(\mathcal{G}_1(L) \cap \text{im}(\alpha)) \cong \mathcal{G}_{\text{Inn}}(L)/\text{im}(\alpha) \), and (b) follows. \( \square \)

3. Proof of Theorem 3. Theorem 3 is proved by a modification of an argument of Bachmuth and Mochizuki.

**Theorem 3.** Let \( G \) be any group admitting a surjective homomorphism \( \eta: G \to \mathbb{Z} \times \mathbb{Z} \). Then \( \text{GL}_2(\mathbb{Z}G) \) is infinitely-generated.
Proof. Let s and t be generators of $\mathbb{Z} \times \mathbb{Z}$, and denote the group ring $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] = \mathbb{Z}[s, s^{-1}, t, t^{-1}]$ by $R$. The homomorphism $\eta$ induces a homomorphism $\beta : \text{GL}_2(\mathbb{Z}G) \to \text{GL}_2(R)$. Let $S = \text{SL}_2(R) \cap \text{im}(\beta)$.

We begin by using an idea from [2] to show that $\text{GL}_2(\mathbb{Z}G)$ is infinitely-generated if $S$ is. There is a short exact sequence

$$1 \to \text{SL}_2(\mathbb{Z}) \to \text{GL}_2(R) \to R^* \to 1$$

where the group of units $R^*$ is generated by $\{-1, s, t\}$. Let $R_0 \subset R^*$ be the subgroup generated by $\{s^2, t^2\}$, which has index 8 in $R^*$, and let $H = \det^{-1}(R_0)$. If $w \in R_0$, then the "positive" square root $w^{1/2}$ is uniquely defined, and $f : H \to \text{SL}_2(R)$ defined by $f(A) = (\det(A))^{-1/2}A$ is a retraction. Let $K = \text{image}(\beta)$. Since $H$ has finite index in $\text{GL}_2(R)$, $K \cap H$ has finite index in $K$. But $f|_{K \cap H}$ retracts $K \cap H$ onto $S$. Therefore, if $S$ is infinitely-generated, then so is $\text{GL}_2(\mathbb{Z}G)$.

The proof that $S$ is infinitely-generated is a minor modification (for the case $P = \mathbb{Z}$) of the argument of §2 of [1], and we use the notation of that paper. Choose $x, y \in G$ with $r/(x) = 5$ and $r/(y) = 1$. Lemma 1 of [1] is replaced by

**Lemma 1'.** $E_2(R)$ is contained in

$$\left(S \cap \text{SL}_2(\mathbb{Z}[s, s^{-1}, t])\right) \ast V \left(S \cap \text{SL}_2(\mathbb{Z}[s, s^{-1}, t])^{[0, 1]}\right),$$

where $V$ is the intersection of the factors.

**Lemma 2'**. Let $\pi$ be a nonunit element of $\mathbb{Z}$. Then, the matrices

$$\begin{bmatrix} 1 & 0 \\ (s - 1)/\pi^i & 1 \end{bmatrix}, \quad i \geq 1,$$

can be chosen as part of a set of double coset representatives of $(S \cap \text{SL}_2(\mathbb{Z}[s, s^{-1}, t]), U)$ in $\text{SL}_2(\mathbb{Q}[s, s^{-1}, T])$.

The proof is unchanged. Alternatively, since

$S \cap \text{SL}_2(\mathbb{Z}[s, s^{-1}, t]) \subseteq \text{SL}_2(\mathbb{Z}[s, s^{-1}, t]),$

Lemma 2 implies Lemma 2'.

The argument continues exactly as in [1]. In the final calculation, one must check that

$$M_i \begin{bmatrix} 1 & \pi^2 t^{-1} \\ 0 & 1 \end{bmatrix} M_i^{-1} = \begin{bmatrix} 1 - \pi^i(s - 1)t^{-1} & \pi^2 t^{-1} \\ -(s - 1)^2 t^{-1} & 1 + \pi^i(s - 1)t^{-1} \end{bmatrix}$$

is in $S$. Let $D_i$ be this matrix, and let $D'_i$ be the matrix

$$D'_i = \begin{bmatrix} 1 - \pi^i y^{-1}(x - 1) & \pi^2 t^{-1} \\ -(x - 1)^2 y^{-1}(x - 1) & 1 + \pi^i(x - 1)y^{-1} \end{bmatrix}$$
with entries in $\mathbb{Z}G$. Then $D'_i$ is invertible with two-sided inverse
\[
\begin{bmatrix}
1 + \pi'y^{-1}(x - 1) & -\pi'^2y^{-1} \\
(x - 1)y^{-1}(x - 1) & 1 - \pi'(x - 1)y^{-1}
\end{bmatrix}
\]
and $\beta(D'_i) = D_i$. □

**REFERENCES**