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OPEN RETRACTIONS ON LOCALLY CONNECTED CONTINUA

S. MIKLOS

ABSTRACT. It is shown that a locally connected continuum X admits open retractions onto each of its subcontinua if and only if X is either an arc or a simple closed curve.

We consider continua X have the following property Γ:

(Γ) X admits open retractions onto each of its subcontinua.

Examples of continua having property Γ are:

1. An arc and a simple closed curve.
2. A symmetrical topologist's sine curve K, i.e., $K = P \cup Q$, where P is the closure of the set $\{(x, y) \in R^2: y = \sin 1/x \text{ and } x \in (0, 1/\pi]\}$, and Q is the image of P under the symmetry with respect to the line $x = 1/\pi$.
3. The simplest indecomposable plane continuum (see [**2**, p. 204]).
4. A solenoid.

Problem. Characterize all continua having the property Γ.

In this paper we obtain a partial solution, for locally connected continua.

THEOREM. *A locally connected continuum has the property* Γ *if and only if it is either an arc or a simple closed curve.*

We need the following lemmas.

LEMMA 1. *Let X be a hereditarily locally connected continuum (i.e., every subcontinuum is locally connected) and let Y be a subcontinuum of X such that $X \setminus Y$ has infinitely many components. Then there exists a subcontinuum Z of X such that $Z \supsetneq Y$, $X \setminus Z$ has infinitely many components, and for every component C of $X \setminus Z$, $Z \setminus \overline{C}$ is connected.*

PROOF. Let K be a component of $X \setminus Y$, and consider the continuum $Y \cup K$. Choose $p \in K$, put $\varepsilon = d(p, Y)$, the distance from p to Y, and let $\mathscr{U}$ be a finite open connected cover of the locally connected continuum $Y \cup K$ (in $Y \cup K$) with mesh $\mathscr{U} < \varepsilon/2$. Let $\{U_1, U_2, \ldots, U_n\}$ be a subcollection of $\mathscr{U}$ covering Y such that every U_i intersects Y. Then the union $U_K = \bigcup\{U_i: i \in \{1, 2, \ldots, n\}\}$ contains Y and is connected. Thus the continuum $\overline{U}_K$ is such that $Y \subsetneq \overline{U}_K \subset (Y \cup K) \setminus \{p\}$ and,

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for each component C of $(Y \cup K) \setminus \overline{U}_K$ we have $U_K \subset \overline{U}_K \setminus \overline{C} \subset \overline{U}_K$, whence we conclude $\overline{U}_K \setminus \overline{C}$ is connected.

Further, let $\{K_t\}_{t \in T}$ be the family of all components of $X \setminus Y$. For each index $t \in T$ we take this set U_{K_t} defined as above, and we define $Z = \bigcup\{\overline{U}_{K_t}: t \in T\}$. Then Z is obviously connected, and by local connectedness of X it is a continuum. Hence we have $Y \subsetneq Z \subset X$ and $X \setminus Z$ has infinitely many components. Let C be any of them. Then there is an index $s \in T$ such that C is a component of $(Y \cup K_s) \setminus \overline{U}_{K_s}$ with $C \subset K_s$ and $\bigcup_{t \in T \setminus \{s\}} \overline{U}_{K_t} \cup U_{K_s} \subset Z \setminus \overline{C} \subset Z$. Thus $Z \setminus \overline{C}$ is connected, and the proof is complete.

LEMMA 2. *Let X be a locally connected continuum, Y a proper subcontinuum of X, and C a component of $X \setminus Y$ such that $Y \setminus \overline{C}$ is connected, and suppose there exists an open retraction $r: X \to Y$. Then $r(\overline{C}) = Y$.*

PROOF. $r(C)$ is a nonempty open subset of Y, since C is a nonempty open subset of X. Thus $r(C) \setminus \overline{C}$ is nonempty open in $Y \setminus \overline{C}$. We have $r(C) \setminus \overline{C} = r(\overline{C}) \setminus \overline{C}$, since $r(\overline{C}) = (r(C) \setminus \overline{C}) \cup (Y \cap \overline{C})$. Thus $r(C) \setminus \overline{C}$ is also closed in $Y \setminus \overline{C}$. Since $Y \setminus \overline{C}$ is connected, we must have $r(C) \setminus \overline{C} = Y \setminus \overline{C}$. Hence

$$r(\overline{C}) = (r(C) \setminus \overline{C}) \cup (Y \cap \overline{C}) = (Y \setminus \overline{C}) \cup (Y \cap \overline{C}) = Y.$$

A continuum is called a Θ-space [**1**] if the complement of every subcontinuum has only finitely many components. It is known that a locally connected continuum is a Θ-space if and only if it is a graph [**1**]. We use this to prove the following

PROPOSITION 1. *Every locally connected continuum with the property Γ is a graph.*

PROOF. Let X be any locally connected continuum having the property Γ. Suppose that X is not a Θ-space. Then there exists a subcontinuum Y of X such that the family of all components of $X \setminus Y$ is infinite. Since X has the property Γ, it is hereditarily locally connected. Thus, by Lemma 1, there exists a subcontinuum Z of X with $Y \subsetneq Z$ such that the family $\mathscr{C}$ of all components of $X \setminus Z$ is infinite, and such that for every member C of $\mathscr{C}$ the set $Z \setminus \overline{C}$ is connected. Since X has the property Γ, there exists an open retraction $r: X \to Z$ from X onto Z. By Lemma 2 the closure of every member of $\mathscr{C}$ is mapped onto Z. By compactness of X, there exists a sequence $\{\overline{C}_i\}$ of closures of members of $\mathscr{C}$, with each C_i contained in a distinct component of $X \setminus Y$, such that $\{\overline{C}_i\}$ converges to a compactum D. Since the components of $X \setminus Y$ are open sets we must have $D \subset Y$. Since $r(\overline{C}_i) = Z$ for each i, $r(D) = Z$. But $r(D) = D \subsetneq Z$, a contradiction. Thus X is a Θ-space and therefore a graph.

PROPOSITION 2. *Every graph with the property Γ is either an arc or a simple closed curve.*

PROOF. Let X be a graph with property Γ. We show that X has no ramification points, and is therefore an arc or a simple closed curve. It is easily seen that X contains an arc α, no point of which is a ramification point, such that $X \setminus \alpha$ is connected. Then $Y = X \setminus \operatorname{int} \alpha$ and $C = \operatorname{int} \alpha$ satisfy the hypotheses of Lemma 2, and every open retraction $r: X \to Y$ must take $\overline{C} = \alpha$ onto Y. Now suppose X has a ramification point y. Then $y \in Y$, hence $y = r(p)$ for some $p \in \alpha$. Since p is a nonramification point, there exists a sufficiently small closed neighborhood V of p in X such that V has at most two boundary points in X, while $r(V)$ is a closed neighborhood of y with more than two boundary points in Y. Thus r must take some interior point of V onto a boundary point of $r(V)$, contradicting the openness of r.

The proof of the theorem follows from Example 1 and the above propositions.

The examples 1, 2, 3 and 4 suggest the following

Question. Is every continuum with the property Γ atriodic?

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Institute of Mathematics of the Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland