\textbf{\omega\text{-}CONNECTED CONTINUA AND JONES' K FUNCTION}

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\textbf{Abstract.} A continuum $X$ is $\omega\text{-}connected$ if for every pair of points $x, y$ of $X$, there exists an irreducible subcontinuum of $X$ from $x$ to $y$ that is decomposable. If $A \subset X$ then $K(A)$ is the intersection of all subcontinua of $X$ that contain $A$ in their interiors. The main theorem shows that if $X$ is an $\omega\text{-}connected$ continuum and $H$ is a connected nowhere dense subset of $X$, then $K(H)$ has a void interior. Several corollaries are established for continua with certain separation properties and a final theorem shows the equivalence of $\omega\text{-}connectedness$ and $\delta\text{-}connectedness$ for plane continua.

Let $X$ be a compact connected metric space (continuum). If for every pair of distinct points $x, y$ of $X$ there exists a subcontinuum $I$ of $X$ irreducible between $x$ and $y$ such that: (1) $I$ is decomposable, then $X$ is $\omega\text{-}connected$ [3]; (2) $I$ contains no indecomposable subcontinuum with nonvoid interior relative to $I$, then $X$ is $\lambda\text{-}connected$ [7]; (3) $I$ is hereditarily decomposable, then $I$ is $\delta\text{-}connected$ [7]; (4) $I$ is an arc, then $I$ is $\alpha\text{-}connected$ (arcwise connected). The first three properties are generalizations of arcwise connectedness and it is clear that $\alpha\text{-}connectedness$ implies $\delta\text{-}connectedness$ which implies $\lambda\text{-}connectedness$ which implies $\omega\text{-}connectedness$. Also if $A \subset X$ let $K(A)$ be the intersection of all subcontinua of $X$ that contain $A$ in their interiors relative to $X$. This concept was introduced by F. B. Jones in [9, Theorem 2]. There the $K$ function is restricted to points (rather than subsets).

In [10] H. E. Schlais proves the following:

\textbf{Theorem.} If $X$ is a hereditarily decomposable continuum, then for every point $x$ of $X$, the interior of $K(x)$ relative to $X$ is void.

C. L. Hagopian [4, Theorem 4] generalized this result in two ways in the following:

\textbf{Theorem.} If $X$ is a $\delta\text{-}connected$ continuum, then for any connected nowhere dense subset $H$ of $X$, the interior of $K(H)$ relative to $X$ is void.

Hagopian then raised the question as to whether his theorem was true for $\lambda\text{-}connected$ continua [6]. The main theorem of this paper is to answer that question in the affirmative by proving that the theorem is true, in fact, for $\omega\text{-}connected$...
continua. As consequences of this theorem, several results concerning monotone, upper-semicontinuous decompositions for continua with certain separation properties are established. Hagopian has also shown [5, Theorem 2] the equivalence of \( \delta \)-connectedness and \( \lambda \)-connectedness for plane continua. A short proof is given at the close of this paper showing that in fact \( \omega \)-connectedness, \( \lambda \)-connectedness and \( \delta \)-connectedness are all equivalent for plane continua.

**Theorem 1.** If \( X \) is an \( \omega \)-connected continuum, then for each connected nowhere dense subset \( H \) of \( X \), the interior of \( K(H) \) relative to \( X \) is void.

**Proof.** Let \( H \) be a connected nowhere dense subset of \( X \) and assume that \( K(H) \) has a nonvoid interior relative to \( X \). Let \( W \) be a nonvoid open subset of \( X \) whose closure is contained in the interior of \( K(H) \). Define \( T \) to be the component of \( X \setminus W \) that contains \( H \) and let \( K = T \cap \text{Bd}(W) \). There exist open sets \( U_0, C_0 \) such that \( \text{diam}(U_0) < 1, K \subset C_0 \subset \text{Cl}(C_0) \subset \text{Int}(K(H) \setminus H), \text{dist}(x, K) < 1 \) for all \( x \in C_0, \text{Cl}(U_0) \subset W \) and \( \text{Cl}(U_0) \cap \text{Cl}(C_0) = \emptyset \).

Observe that the component \( P \) of \( X \setminus U_0 \) that contains \( H \) (and hence contains \( T \)) cannot contain \( C_0 \). To see this, suppose \( C_0 \subset P \). Since \( H \subset \text{Int}(P) \) (for otherwise \( U_0 \subset K(H) \)), there exists a sequence of points \( x_1, x_2, \ldots \) in \( X \setminus (P \cup U_0) \) converging to a point of \( H \). For each positive integer \( i \), define \( S_i \) to be the \( x_i \)-component of \( X \setminus U_0 \). Then \( S = \text{lim sup} S_i \) is a continuum in \( X \setminus U_0 \) that intersects both \( H \) and \( \text{Cl}(U_0) \). Since \( C_0 \subset P \), for each positive integer \( i \), \( S_i \cap C_0 \neq \emptyset \), hence \( S \cap C_0 = \emptyset \). But \( K \subset C_0 \) and \( S \subset T = \emptyset \) so \( S \cap W = \emptyset \). This means that \( S \subset T \) and hence that \( T \cap \text{Cl}(U_0) \neq \emptyset \) since \( S \cap \text{Cl}(U_0) \neq \emptyset \). This is a contradiction since \( T \subset X \setminus W \) and \( \text{Cl}(U_0) \subset W \). Therefore \( C_0 \) is not a subset of \( P \).

Thus we have \( X \setminus U_0 = L_0 \cup R_0 \), a separation, such that \( K \subset P \subset L_0 \) and \( R_0 \cap C_0 \neq \emptyset \). Define \( V_0, C_1 \) to be open sets such that \( \text{diam}(V_0) < 1, K \subset C_1 \subset \text{Cl}(C_1) \subset L_0 \cap C_0 \) and \( \text{Cl}(V_0) \subset R_0 \cap C_0 \). Now replace \( U_0 \) and \( C_0 \) by \( V_0 \) and \( C_1 \), respectively, and follow the argument of the last paragraph word for word down to the final two sentences. This gives a contradiction since \( T \subset L_0 \cap (X \setminus W) \) and \( \text{Cl}(V_0) \subset R_0 \). Therefore \( C_1 \) is not a subset of \( P \), the component of \( X \setminus V_0 \) that contains \( H \). Therefore \( X \setminus V_0 = L_1 \cup R_1 \), a separation, such that \( K \subset L_1 \cap C_1 \) and \( R_1 \cap C_1 \neq \emptyset \). Note that \( L_1 \cap U_0 \neq \emptyset \) and \( R_1 \cap U_0 \neq \emptyset \).

We proceed by induction. Assume that open sets \( U_i, V_i, C_{2i}, C_{2i+1} \) of \( X \) have been defined for \( 0 \leq i \leq n - 1 \) such that

1. \( \text{diam}(U_i) < 1/2^i, \text{diam}(V_i) < 1/2^i \);
2. \( X \setminus U_i = L_{2i} \cup R_{2i}, \) a separation;
3. \( K \subset C_{2i+1} \subset \text{Cl}(C_{2i+1}) \subset L_{2i} \cap C_{2i}, \text{Cl}(V_i) \subset R_{2i} \cap C_{2i}, \text{dist}(x, K) < 1/2^i \) for all \( x \in C_{2i} \);
4. \( X \setminus V_i = L_{2i+1} \cup R_{2i+1}, \) a separation;
5. \( K \subset L_{2i+1} \cap C_{2i+1}, R_{2i+1} \cap C_{2i+1} \neq \emptyset ; \) and
6. \( \text{Cl}(U_{i+1}) \subset U_i \cap R_{2i+1} \) for \( i < n - 1 \).

Let \( C_{2n} \) be an open set such that \( K \subset C_{2n} \subset \text{Cl}(C_{2n}) \subset L_{2n-1} \cap C_{2n-1} \) and \( \text{dist}(x, K) < 1/2^n \) for all \( x \in C_{2n} \). By (2)--(5), \( R_{2n-1} \cap U_{n-1} \neq \emptyset \), so let \( U_n \) be an open set such that \( \text{Cl}(U_n) \subset R_{2n-1} \cap U_{n-1} \) and \( \text{diam} U_n < 1/2^n \). Our previous
argument yields the fact that $X \setminus U_n = L_{2n} \cup R_{2n}$, a separation, such that $K \subset L_{2n} \cap C_{2n}$ and $R_{2n} \cap C_{2n} \neq \emptyset$. Define $V_n$, $C_{2n+1}$ to be open sets such that $K \subset C_{2n+1} \subset \text{Cl}(C_{2n+1}) \subset L_{2n} \cap C_{2n}$ and $\text{Cl}(V_n) \subset R_{2n} \cap C_{2n}$. As before we have $X \setminus V_n = L_{2n+1} \cup R_{2n+1}$, a separation, such that $K \subset L_{2n+1} \cap C_{2n+1}$ and $R_{2n+1} \cap C_{2n+1} \neq \emptyset$.

Let $\{x\} = \bigcap U_i$ and (taking subsequences if necessary) let $y = \lim V_i$. Note that $y \notin K$ since $K = \bigcap C_i$ and $V_i \subset C_{2i}$ for $i = 0, 1, \ldots$. By hypothesis there exists an irreducible subcontinuum $I$ of $X$ from $x$ to $y$ that is decomposable. Set $I = I_x \cup I_y$, where $I_x$ and $I_y$ are continua such that $x \in I_x \setminus I_y$, $y \in I_y \setminus I_x$. Let $k$ be an integer such that $U_k \cap I_y = \emptyset$ and $V_k \cap I_x = \emptyset$. Then $I_x \subset R_{2k+1}$ since $x \in R_{2k+1}$. Also $I_y \subset L_{2k}$ since $y \in K$ and $K \subset \bigcap L_i$. Then since $I_y \subset L_{2k}$, $V_k \subset R_{2k}$ and $y \in L_{2k+1}$, it follows that $I_y \subset L_{2k+1}$. So we have $I = I_x \cup I_y \subset L_{2k+1} \cup R_{2k+1}$, a separation, with $x \in R_{2k+1}$ and $y \in L_{2k+1}$. This is a contradiction, so the assumption that $K(H)$ has a nonvoid interior is false. This completes the proof of the theorem.

A $\theta_n$-continuum ($\theta$-continuum) is a continuum such that every subcontinuum separates it into at most $n$ components (a finite number of components). The following theorem has been established for $\theta_n$-continua in [2, Theorem 2] and for $\theta$-continua in [1, Theorem 2].

**Theorem.** Let $X$ be a $\theta_n$-continuum or a $\theta$-continuum. Then $X$ admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors, and which is unique and minimal with respect to the property that the quotient space is a finite graph if and only if whenever $H$ is a nowhere dense subcontinuum of $X$, it follows that $K(H)$ is nowhere dense.

As a consequence of this theorem and Theorem 1, the next result follows immediately.

**Theorem 2.** Let $X$ be a $\theta_n$-continuum or a $\theta$-continuum and let $X$ be $\omega$-connected. Then $X$ admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors and which is unique and minimal with respect to the property that the quotient space is a finite graph.

Specifically for $\theta_1$-continua (continua for which the complement of every subcontinuum is connected), Theorem 3 provides a generalization of Hagopian’s decomposition theorem for $\delta$-connected $\theta_1$-continua [4, Theorem 5].

**Theorem 3.** Let $X$ be an $\omega$-connected $\theta_1$-continuum. Then $X$ admits a monotone, upper-semicontinuous decomposition, the elements of which have void interiors and which is unique and minimal with respect to the property that the quotient space is a simple closed curve.

The equivalence of $\delta$-connectedness and $\omega$-connectedness for plane continua is observed as a final result. The proof consists of combining the following two theorems.

**Theorem (Hagopian [4, Theorem 2]).** A plane continuum is $\delta$-connected if and only if it cannot be mapped continuously onto $D$, the Knaster plane indecomposable continuum with one endpoint.
**Theorem (Krasinkiewicz and Minc [8, Theorem 10]).** Let \( X \) be a plane continuum which can be mapped continuously onto an indecomposable continuum. Then there exist two points \( x, y \in X \) such that \( X \) contains exactly one subcontinuum that is irreducible between \( x \) and \( y \) and this subcontinuum is indecomposable.

**Theorem 4.** If \( X \) is an \( \omega \)-connected plane continuum then \( X \) is \( \delta \)-connected.

**Proof.** Suppose \( X \) is not \( \delta \)-connected. According to Hagopian’s theorem \( X \) can be mapped continuously onto the Knaster indecomposable continuum \( D \). Then by the theorem of Krasinkiewicz and Minc, there are two points \( x, y \in X \) such that there is only one subcontinuum that is irreducible between \( x \) and \( y \) and it is indecomposable. But this is a contradiction to the \( \omega \)-connectedness of \( X \). Hence \( X \) is \( \delta \)-connected.

**References**


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