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ON COLLECTIONWISE NORMALITY OF PRODUCT SPACES. I

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ABSTRACT. In this paper the following result will be obtained: Let X be a collectionwise normal Σ-space (in the sense of Nagami [**9**]) and Y a paracompact first countable P-space (in the sense of Morita [**8**]). Then $X \times Y$ is collectionwise normal.

1. Introduction. Throughout this paper all spaces are Hausdorff spaces.

On collectionwise normality of a product space $X \times Y$, the following theorems are known.

(I) (KOMBAROV [**5**]). *Let X be a normal countably compact space and Y a paracompact sequential space. Then $X \times Y$ is collectionwise normal.*

(II) (YAJIMA [**14**]). *Let X be a collectionwise normal space which has a σ-closure preserving closed cover by countably compact sets and Y a paracompact first countable space. Then $X \times Y$ is collectionwise normal.*

We shall consider another condition of X and Y such that $X \times Y$ is collectionwise normal. The following are known.

(III) (NAGAMI [**9**]). *Let X be a paracompact Σ-space and Y a paracompact P-space. Then $X \times Y$ is paracompact.*

(IV) [**2**]. *Let X be a normal M-space and Y a paracompact first countable P-space. Then $X \times Y$ is normal.*

(V) [**2**]. *There exist a normal σ-space X and a compact first countable space Y such that $X \times Y$ is not normal* (*see* §3, *Example* 3).

In this paper we shall prove the following theorem which contains (IV).

THEOREM. *Let X be a collectionwise normal Σ-space and Y a paracompact first countable P-space. Then $X \times Y$ is collectionwise normal.*

The definitions of Σ-spaces are due to Nagami [**9**], P-spaces and M-spaces are due to Morita [**8**], and σ-spaces are due to Okuyama [**10**].

2. Proof of Theorem. For the proof, we shall use the following facts.

FACT 1. Let $\mathfrak{A} = \{A_\gamma | \gamma \in \Gamma\}$ be a discrete collection of closed subsets of X. If there exists a normal open cover of X each of whose members meets at most

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one A_γ, then there are open sets H_γ of X such that $H_\gamma \supset A_\gamma$ for each $\gamma \in \Gamma$ and $H_\gamma \cap H_\mu = \emptyset$ if $\gamma \neq \mu$.

Fact 1 is well known.

FACT 2 [**9**, Lemma 1.4]. Let X be a Σ-space. Then X has a Σ-net $\{\mathfrak{F}_n | n = 1, 2, \dots\}$ which satisfies the following conditions:

(N_1) $\mathfrak{F}_n = \{F(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Xi\}$.

(N_2) Every $F(\alpha_1, \dots, \alpha_n) = \bigcup \{F(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) | \alpha_{n+1} \in \Xi\}$.

(N_3) For every $x \in X$, there exists a sequence $\alpha_1, \alpha_2, \dots$ such that $\{F(\alpha_1, \dots, \alpha_n) | n = 1, 2, \dots\}$ is a net of $C(x)$.

Here $C(x) = \bigcap \{C(x, \mathfrak{F}_n) | n = 1, 2, \dots\}$, $C(x, \mathfrak{F}_n) = \bigcap \{F | x \in F \in \mathfrak{F}_n\}$.

PROOF OF THEOREM. [1] This proof is a modification of that of (III) (Theorem 4.1 in [**9**]). Let X be a collectionwise normal Σ-space and Y a paracompact first countable P-space. Let $\{\mathfrak{F}_n | n = 1, 2, \dots\}$ be a Σ-net of X satisfying the conditions (N_1)–(N_3) in Fact 2. Since $\mathfrak{F}_n$ is a locally finite closed cover of X and X is strongly normal, by Katětov [**4**], there exists a locally finite cozero-set cover $\mathfrak{H}_n = \{H(\alpha_1, \dots, \alpha_n) | \alpha_1, \dots, \alpha_n \in \Xi\}$ such that

$$F(\alpha_1, \dots, \alpha_n) \subset H(\alpha_1, \dots, \alpha_n) \quad \text{for each } \alpha_1, \dots, \alpha_n \in \Xi.$$

Let $\mathfrak{A}$ be a discrete family of closed subsets of $X \times Y$. Let $\mathfrak{W}(\alpha_1, \dots, \alpha_n) = \{U_\lambda \times V_\lambda \ (\neq \emptyset) \ | \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$ be the maximal collection satisfying the following conditions:

(1) Each U_λ is a finite union of cozero-sets $\{U_{\lambda,j} | 1 \leq j \leq m(\lambda)\}$ of X such that

$$F(\alpha_1, \dots, \alpha_n) \subset U_\lambda \subset H(\alpha_1, \dots, \alpha_n).$$

(2) Each V_λ is an open set of Y.

(3) Each memeber of $\mathfrak{J}_\lambda = \{U_{\lambda,j} \times V_\lambda | 1 \leq j \leq m(\lambda)\}$ meets at most one member of $\mathfrak{A}$.

Let us put $V(\alpha_1, \dots, \alpha_n) = \bigcup \{V_\lambda | \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$. Then $V(\alpha_1, \dots, \alpha_n) \subset V(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Xi$. Since Y is a P-space, for each $\alpha_1, \dots, \alpha_n \in \Xi$, there exists a closed set $K(\alpha_1, \dots, \alpha_n)$ of Y such that

(4) $K(\alpha_1, \dots, \alpha_n) \subset V(\alpha_1, \dots, \alpha_n)$.

(5) If $\bigcup_{n=1}^{\omega} V(\alpha_1, \dots, \alpha_n) = Y$, then $\bigcup_{n=1}^{\omega} K(\alpha_1, \dots, \alpha_n) = Y$, where ω denotes the first infinite ordinal.

Since Y is paracompact, for each $\alpha_1, \dots, \alpha_n \in \Xi$, there exists a locally finite collection $\{V'_\lambda | \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$ of cozero-sets in Y such that:

(6) $V'_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda(\alpha_1, \dots, \alpha_n)$.

(7) $K(\alpha_1, \dots, \alpha_n) \subset \bigcup \{V'_\lambda | \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$.

Let us put $\mathfrak{G}_n = \{U_{\lambda,j} \times V'_\lambda | \lambda \in \Lambda(\alpha_1, \dots, \alpha_n),\ \alpha_1, \dots, \alpha_n \in \Xi,\ 1 \leq j \leq m(\lambda)\}$ for each $n = 1, 2, \dots$, and put $\mathfrak{G} = \bigcup \{\mathfrak{G}_n | n = 1, 2, \dots\}$.

Then we have:

(8) Each $\mathfrak{G}_n$ is locally finite in $X \times Y$.

(9) Each member of $\mathfrak{G}$ meets at most one member of $\mathfrak{A}$.

(10) Each member of $\mathfrak{G}$ is a cozero-set in $X \times Y$.

(11) $\mathfrak{G}$ is a cover of $X \times Y$.

(8)–(10) are clear.

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[1] The author first proved this theorem by another method. Y. Yajima pointed out that we can give a simpler proof by modifying the proof of (III).

PROOF OF (11). Let $(x,y) \in X \times Y$ be an arbitrary element. Let $\alpha_1, \ldots, \alpha_n, \ldots \in \Xi$ be elements such that $\{F(\alpha_1, \ldots, \alpha_n) | n = 1, 2, \ldots\}$ is a net of $C(x)$. Then we have $\bigcup_{n=1}^{\omega} V(\alpha_1, \ldots, \alpha_n) = Y$. To show this, let y' be an arbitrary element of Y. Then, since $C(x)$ is countably compact [**9**] and the family $\mathfrak{A}$ is discrete, $\{A_\xi \in \mathfrak{A} | (C(x) \times \{y'\}) \cap A_\xi \neq \emptyset\}$ is finite. Therefore, by using the first countability of Y, there is a finite family $\{M_j | j = 1, 2, \ldots, k\}$ of open sets in X and an open set G in Y such that:

(12) $C(x) \subset \bigcup_{j=1}^{k} M_j$, $y' \in G$.

(13) Each $M_j \times G$ meets at one member of $\mathfrak{A}$.

By Lemma 2.1 in [**14**], there are cozero-sets M_j' in X such that $M_j' \subset M_j$ and $C(x) \subset \bigcup_{j=1}^{k} M_j'$. Then $F(\alpha_1, \ldots, \alpha_i) \subset \bigcup_{j=1}^{k} M_j'$ for some i. Let us put $U_j = M_j' \cap H(\alpha_1, \ldots, \alpha_i)$. Then U_j are cozero-sets in X and $(\bigcup_{j=1}^{k} U_j) \times G \in \mathfrak{W}(\alpha_1, \ldots, \alpha_i)$ by the maximality of $\mathfrak{W}(\alpha_1, \ldots, \alpha_i)$. Thus $y' \in V(\alpha_1, \ldots, \alpha_i)$.

Therefore we have $\bigcup_{n=1}^{\omega} K(\alpha_1, \ldots, \alpha_n) = Y$ by (5). Hence $y \in K(\alpha_1, \ldots, \alpha_n)$ for some n. By (7), $y \in V_\lambda'$ for some $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)$. Then

$$(x,y) \in C(x) \times \{y\} \subset F(\alpha_1, \ldots, \alpha_n) \times V_\lambda' \subset U_\lambda \times V_\lambda'.$$

Since $x \in U_{\lambda,j}$ for some $j \leq m(\lambda)$, $(x,y) \in U_{\lambda,j} \times V_\lambda' \in \mathfrak{G}_n \subset \mathfrak{G}$.

By (8)–(11), $\mathfrak{G}$ is a normal open cover of $X \times Y$, each of whose members meets at most one element of $\mathfrak{A}$. By Fact 1, there exists a disjoint family $\{H_A | A \in \mathfrak{A}\}$ of open sets in $X \times Y$ such that $H_A \supset A$ for each $A \in \mathfrak{A}$. Hence $X \times Y$ is collectionwise normal. The proof of the Theorem is complete.

3. Remarks and examples.

REMARK 1. Our Theorem is neither contained in (I) nor (II) in §1. In fact, let X be the space of irrationals of R with the euclidean topology where R is the real line and Y *the Michael line* [**6**]; then Y is a paracompact first countable space and $X \times Y$ is not normal [**6**]. Therefore X does not satisfy the condition in (II). Also X is not countably compact. But X is a collectionwise normal Σ-space.

Moreover, this example shows that the condition "*Y is a P-space*" cannot be dropped in the Theorem

REMARK 2. We cannot weaken the condition "*Y is first countable*" to the condition "*for each $y \in Y$ is a G_δ-set*". In fact the following example exists.

EXAMPLE 1 [**2**]. Let $X = [0, \omega_1) = \{\alpha | \alpha < \omega_1\}$ with the order topology where ω_1 is the first uncountable ordinal. Then it is well known that X is a normal countably compact space. Let $Y = ([0, \omega) \times [0, \omega_1)) \cup \{(\omega, \omega_1)\}$ with the topology as follows: $\{([\alpha, \omega] \times [\beta, \omega_1]) \cap Y | \alpha < \omega, \beta < \omega_1\}$ is a neighborhood base of (ω, ω_1) and for each $y \in Y - \{(\omega, \omega_1)\}$, y is an isolated point of Y. Then Y is a paracompact perfectly normal space but Y is not first countable. Also $X \times Y$ is not normal [**2**].

REMARK 3. *The paracompactness of* Y cannot be weakened to the condition "*collectionwise normal*". In fact the following example exists.

EXAMPLE 2. There exists a compact space X and a collectionwise normal perfectly normal first countable space Y such that $X \times Y$ is not normal. Let Y be the space constructed by R. Pol in [**11**]. Then Y has the above properties, but Y is not paracompact. Therefore, by the Theorem of Tamano [**13**], there exists a compact space X such that $X \times Y$ is not normal.

REMARK 4. The condition "*X is a Σ-space*" cannot be replaced by the condition "*X is a P-space*". In fact, let X be the Sorgenfrey line [**12**]; then X is a paracompact first countable P-space such that $X^2 = X \times X$ is not normal.

EXAMPLE 3 [**2**]. There exists a normal σ-space X and a compact first countable space Y such that $X \times Y$ is not normal. Let Y be the "*two arrow space*" i.e., let E be the unit square with lexicographic order (cf. [**7**, Example 10.4]). Let $Y = (\{y \mid 0 < y \leq 1\} \times \{0\}) \cup (\{y \mid 0 \leq y < 1\} \times \{1\})$ with the subspace topology of E. Then Y is a compact first countable space. Let X be the space in *Bing's Example* H [**1**] constructed by a suitable set P. Then X is a normal σ-space and $X \times Y$ is not normal (this follows from the proof of Theorem 1 in [**3**] because Y is separable and not metrizable; also see p. 6 in [**2**]).

REMARK 5. This author does not know whether we can generalize the condition "*Y is first countable*" to "*Y is sequential*" or not.

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