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A CHARACTERIZATION OF ALGEBRAS OF INVARIANT-COINVARIANT MODULE TYPE

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ABSTRACT. K. R. Fuller has characterized rings of left invariant module type. An algebra is said to be of invariant-coinvariant module type if each of its indecomposable modules is quasi-injective or quasi-projective. In this note we shall give a characterization of algebras of invariant-coinvariant module type, which distinguishes this class from that of algebras of local-colocal type. It seems of interest that the distributivity of second radicals of primitive ideals appears in our characterization.

Throughout, A denotes an artinian ring with the radical N possessing a unit element 1, and an algebra means a finite dimensional algebra over a field P. All modules considered are unitary A-modules. For an A-module M if the socle (resp. top) of M is simple, M is said to be a colocal (resp. local) module. We shall say that A is of left colocal (resp. local-colocal) type if every indecomposable left A-module is colocal (resp. local or colocal).

A ring A is said to be of left invariant module type if every indecomposable left A-module is quasi-injective. Wu and Jans [**8**] showed that an indecomposable quasi-injective (resp. quasi-projective) module is colocal (resp. local). Thus, as pointed out by Dickson and Fuller [**2**], rings of left invariant module type are of left colocal type. Extending a result on algegras in [**2**] to artinian rings, Fuller [**3**] characterized rings of left invariant module type as those of left colocal type having the property that any primitive right ideal is distributive (i.e. its submodules form a distributive lattice (cf. Camillo [**1**])). In this note we shall say an algebra A is of invariant-coinvariant (module) type if every indecomposable A-module is quasi-injective or quasi-projective. In [**7**] Tachikawa determined the structure of algebras of local-colocal type ([**7**, Theorem 4.3]). Using his structure theorem, it is not hard to show that any algebra A of local-colocal type is of invariant-coinvariant type in case $N^2 = 0$. So arises the question whether or not the class of algebras of invariant-coinvariant type is properly contained in that of algebras of local-colocal type. Our theorem distinguishes algebras of invariant-coinvariant type from those of local-colocal type, and is stated as follows.

THEOREM. *An algebra A is of invariant-coinvariant type if and only if the following conditions are satisfied.*

(I) *A is of local-colocal type.*

(II) *Each of N^2e and eN^2 is distributive for any primitive idempotent e in A.*

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Before proving this theorem we note the following. Let A be an algebra of local-colocal type, and M_i $(i = 1, 2)$ local left A-modules with $NM_i = K_i \oplus L_i$ $(i = 1, 2)$ where K_i and L_i $(i = 1, 2)$ are serial (i.e. its composition series is unique). Then it follows from the proof of Fuller [**4**, Theorem 2.3] and Tachikawa [**7**, Proposition 3.4] that if $M_1/K_1 \simeq M_2/K_2$ and $M_1/L_1 \simeq M_2/L_2$, then $M_1 \simeq M_2$.

PROOF OF THEOREM. *Only if part*: It is necessary only to show that condition (II) holds. Suppose that eN^2 is not distributive for some primitive idempotent e. Then by [**1**, Theorem 1] we can find a submodule X of eN^2 such that the socle of a factor module eN^2/X is not square free. Then since the socle of eN/X is not simple, we have $eN/X = L_1 \oplus L_2$ where L_1 and L_2 are serial by [**7**, Theorem 4.3]. Clearly it is satisfied that the length $|L_1|, |L_2| \geq 2$ and the socles $S(L_1)$ and $S(L_2)$ are isomorphic. Let K_1 and K_2 be the annihilators of L_1 and L_2 in (eA/X^*) respectively, where $(eA/X)^*$ denotes the P-dual module of eA/X. Then K_1 and K_2 are serial modules with $|K_1|, |K_2| \geq 3$ and $K_1/NK_1 \simeq K_2/NK_2$ since $K_1 \simeq ((eA/X)/L_1)^*$ and $K_2 \simeq ((eA/X)/L_2)^*$. Moreover it holds that $K_1 + K_2 = (eA/X)^*$ and $K_1 \cap K_2 = S((eA/X)^*)$.

Assume that $|K_1| > |K_2|$. Let $t_1 = ft_1$ and $t_2 = ft_2$ be generators of K_1 and K_2 respectively, where f is a primitive idempotent. We note that Af is serial by [**7**, Theorem 4.3]. Put $t = t_1 + t_2$; then $At \simeq K_1$. Since $At \cap S^2(K_2)$ is simple where $S^2(K_2)$ is the second socle of K_2 (i.e. $S^2(K_2)/S(K_2) = S(K_2/S(K_2))$), considering the dual of the above note we have $K_1 + S^2(K_2) \simeq At + S^2(K_2)$. Obviously $t \notin K_1 + S^2(K_2)$, and hence $K_1 + S^2(K_2) \neq At + S^2(K_2)$. Therefore $K_1 + S^2(K_2)$ is not invariant by an endomorphism of its injective hull, and hence not quasi-injective by [**5**, Theorem 1.1]. So we obtain an indecomposable A-module which is neither quasi-injective nor quasi-projective. This contradicts the assumption that A is of invariant-coinvariant type.

Next if $|K_1| = |K_2|$, then we have $K_1 + S^2(K_2) \simeq K_2 + S^2(K_1)$ by the note. Since it is clear that $K_1 + S^2(K_2) \neq K_2 + S^2(K_1)$, also in this case $K_1 + S^2(K_2)$ is not quasi-injective. Similarly as the above we have a contradiction.

If part: Indecomposable modules over an algebra of local-colocal type are classified into the next four types: (i) serial, (ii) not serial, simple top and simple socle, (iii) simple top and not simple socle, (iv) simple socle and not simple top. A-modules of type (i) are quasi-injective or quasi-projective by [**4**, Proposition 2.6] and those of type (ii) are projective (and at the same time injective by [**7**, Proposition 3.1]). Besides A-modules of type (iii) and type (iv) are P-dual modules of each other. Thus it is sufficient to show that a left A-module Ae/Y of type (iii) is quasi-projective, where e is a primitive idempotent and Y is a submodule of Ae. We note that $Y \subset N^2e$ because Ae/Y is not serial.

At first let $Ne = Q \oplus R$ where Q and R are serial. Then $N^2e = NQ \oplus NR$, and there does not exist any common composition factor of NQ and NR by the assumption (II). Therefore every submodule of N^2e coincides with $N^iQ \oplus N^jR$ for some $i, j \geq 1$. It follows that Y is invariant by all endomorphisms of Ae. Thus Ae/Y is quasi-projective.

Next if it is not the case, then Ae has the simple socle S and $Y \supset S$. From the fact that Ne/S is a direct sum of two serial modules, similarly as the above we can show that Y/S is invariant by all endomorphisms of Ae/S. Since Ae/S

is quasi-projective, it follows that also $Ae/Y \simeq (Ae/S)/(Y/S)$ is quasi-projective. This completes the proof.

EXAMPLE. Let A be an algebra of matrices of the form

$$\begin{pmatrix} x & 0 & 0 & 0 \\ a & x & 0 & 0 \\ d & b & y & 0 \\ f & e & c & z \end{pmatrix}$$

with entries in a field P. Then A is an algebra of right colocal type. Let e_{ij} be the ijth matrix unit, and $e = e_{11} + e_{22}$. As easily seen, $S(N^2e) \simeq Af/Nf \oplus Af/Nf$, where $f = e_{33}$. So A is not of invariant-coinvariant type.

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