A NOTE ON MAXIMAL OPERATORS
AND REVERSIBLE WEAK TYPE INEQUALITIES

M. A. LECKBAND

ABSTRACT. A class of maximal operators is shown to satisfy a weak type inequality and a corresponding converse inequality. The results are applicable to a fractionally iterated Hardy-Littlewood maximal operator.

1. In this paper we are interested in studying a class of maximal operators that satisfy a weak type inequality that is in some sense reversible. Let $Mf$ denote the Hardy-Littlewood maximal operator of a function $f$ in $L^1(\mathbb{R}^n)$. As is well known, for instance see [1], for suitable constants we have the weak type inequality

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{A_n}{\lambda} \int_{\{|f| > \lambda / A_n\}} |f(x)| dx$$

and the converse inequality

$$\frac{B_n}{\lambda} \int_{\{|f| > \lambda / B_n\}} |f(x)| dx \leq |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}|.$$

Such a converse inequality naturally indicates how good the corresponding weak type inequality is. However, the motivation for the results in this paper is due to an application found in [4 and 5]. Knowing the weak type inequality and its converse for the Hardy-Littlewood maximal operator, an important formula for the $N$th iteration of the maximal operator was obtained. That is, $M$ applied to $f$ $N$ times is pointwise comparable to

$$\sup_{x \in Q} \frac{1}{|Q|} \int_0^{|Q|} (f \cdot \chi_Q)^*(t) \frac{\log^{N-1}|Q|}{(N-1)!} \left(\frac{|Q|}{t}\right) dt,$$

where $(f \cdot \chi_Q)^*$ is the nonincreasing rearrangement of $f$ restricted to the cube $Q$. As is shown in Theorem 3, this new operator satisfies a weak type inequality that has a converse. We replace $(1/(N-1)! \log^{N-1}|Q|/t)$ by a more general function and ask for weak type inequalities that have a converse result.

Let $\Phi : (1, \infty) \to [0, \infty)$ be measurable with $\int_0^1 \Phi(1/t) dt = 1$. Let $f^*$ be the nonincreasing rearrangement of $f$, i.e.,

$$f^*(t) = \inf\{s : |\{|f| > s\}| \leq t\}.$$

We define the $\Phi$-maximal operator $M_\Phi f$ as

$$M_\Phi f(x) = \sup \int_0^1 (f \cdot \chi_Q)^*(|Q/t|) \Phi \left(\frac{1}{t}\right) dt,$$
where the sup is extended over all cubes $Q$ with center $x$ and $\chi_Q$ is the characteristic function of $Q$.

The main results, Theorems 1 and 2, are proven for $\Phi$ which is either nonincreasing or nondecreasing, $t\Phi(t)$ nondecreasing, and for which there are positive constants $C_1$, $C_2$, $0 < \alpha < 2$, $0 < \beta < 1$, such that $C_1 t^\alpha \geq t\Phi(t) \geq C_2 t^\beta$, for $1 \leq t < \infty$. Equivalent maximal operators where $\Phi(t)$ is nondecreasing have been studied (see [3]). For this reason perhaps the more interesting weak type estimates in Theorem 1 occur for $\Phi$ nonincreasing. Theorem 2 contains the converse inequalities to Theorem 1. As a natural application, we consider the iterated Hardy-Littlewood maximal operator. Our results in Theorems 1 and 2 allow us to consider the case $\Phi(1/t) = \log^a(e/t)$, where $a$ is a given real number. We list the resulting corollaries as Theorem 3.

2. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a measurable function, let

$$\lambda_f(s) = \{|x| : |f(x)| > s\},$$

and

$$f^*(t) = \inf\{s: \lambda_f(s) \leq t\}.$$

For a reference on the properties of $f^*(t)$ see [2]. Let $\Phi: (1, \infty) \to [0, \infty)$ be a nonincreasing (nondecreasing), continuous function such that $\int_0^1 \Phi(1/t) \, dt = 1$. We also require that $t\Phi(t)$ be nondecreasing and that there be positive constants $C_1, C_2$, $0 < \alpha < 2$, and $0 < \beta < 1$ with $C_1 t^\alpha \geq t\Phi(t) \geq C_2 t^\beta$, for $1 \leq t < \infty$. We define the $\Phi$-maximal operator of $f$ as

$$M_{\Phi} f(x) = \sup_Q \int_0^1 (f \cdot \chi_Q)^* \left(\frac{|Q| t}{t}\right) \Phi\left(\frac{1}{t}\right) \, dt,$$

where the sup is extended over all cubes $Q$ with center $x$, and $\chi_Q$ is the characteristic function of $Q$.

REMARK 1. If $f$ is not measurably constant, i.e., $|\{x: |f(x)| = a\}| = 0$, for all $a > 0$, then $M_{\Phi} f$ can be computed as

$$M_{\Phi} f(y) = \sup \frac{1}{|Q|} \int_Q |f(x)| \Phi\left(\frac{|Q|}{\rho_{Q,f}(x)}\right) \, dx,$$

where the sup is extended over all cubes $Q$ with center $y$ and

$$\rho_{Q,f}(x) = \inf\{t: x \in \{z: |f(z)| \geq (f \cdot \chi_Q)^*(t)\}\}.$$

We use the convention that $A$, $B$, and $C$ denote constants depending only upon the dimension $n$ and $\Phi$. Any further dependence will be signified by a subscript.

The following lemma contains the inequalities that we will use to show the operator $M$ satisfies a weak type inequality.

LEMMA 1. Let $\lambda > 0$ and $Q \subset \mathbb{R}^n$ such that

$$\lambda \leq \int_0^1 (f \cdot \chi_Q)^* \left(\frac{|Q| t}{t}\right) \Phi\left(\frac{1}{t}\right) \, dt.$$

Then if $\Phi$ is nonincreasing we have

$$1 \leq \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} C \frac{|f(x)|}{\lambda} \Phi\left(\left(C \frac{|f(x)|}{\lambda}\right)^\beta\right) \, dx.$$
For $\Phi$ nondecreasing and $\eta > 1/(2 - \alpha)$, there is a constant $C_\eta > 0$ such that

$$1 \leq \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} C_\eta \frac{|f(x)|}{\lambda} \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \, dx.$$  

**Proof.** Since $\Phi$ is normalized, that is $\int_0^1 \Phi(1/t) \, dt = 1$, we have

$$\frac{1}{2} \leq \int_{[0,1] \cap \{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\}} \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \, dt.$$  

First we do the case where $\Phi$ is nondecreasing. For $0 < \delta < 1$ let

$$J_\delta = \left\{ t \in (0, 1] : \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \leq \frac{1}{4} \left( 1 - \delta \right) \right\}.$$  

Then

$$\frac{1}{2} \leq \int_{\{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\} \setminus J_\delta} \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \, dt + \frac{1}{4} |J_\delta|,$$

or

$$\frac{1}{4} \leq \int_{\{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\} \setminus J_\delta} \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \, dt.$$  

Let $t \in \{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\} \setminus J_\delta$. We use the upper bound on $t \Phi(t)$ to estimate

$$\left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \geq \frac{1 - \delta}{4 \Phi(1/t) t^\delta} \geq \frac{1 - \delta}{4C_1} t^{\alpha - (\delta + 1)}.$$  

We choose $\delta$ such that $\alpha - (\delta + 1) < 0$, obtaining

$$\left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \geq \frac{1}{t}.$$  

where

$$\eta = \frac{1}{1 - \delta + 1} \quad \text{and} \quad C_\eta = \max \left( 4, \left( \frac{4C_1}{1 - \delta} \right) \right).$$  

Since $\Phi$ is nondecreasing we have

$$\frac{1}{4} \leq \int_{\{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\}} \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \, dt$$

$$= \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} \left( \frac{f(x)}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \, dx.$$  

Note that the continuity of $t \Phi[C_\eta t^n]$ gives us the above equality.

Now let $\Phi$ be nonincreasing. We may assume $C_2 < \frac{1}{2}$, where $t \Phi(t) > C_2 t^\beta$, and

$$\int_{\{t (f \cdot x_Q) \ast (|Q| t) > \lambda/2\}} \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \left( \frac{f \cdot x_Q}{\lambda} \right) \Phi \left( \frac{1}{t} \right) \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \, dt < C_2^{2/\beta}.$$  

If not, let $C = C_2^{-2/\beta}$ and we are done, noting that $\beta < 1$ and $\Phi$ is nondecreasing. With the above we use the lower bound for $t \Phi(t)$ and the property that $t \Phi(t)$ is nondecreasing to estimate

$$C_2 \left[ \left( \frac{f \cdot x_Q}{C_2^{2/\beta} \lambda} \right) \right]^\beta \leq \frac{1}{t} \int_0^t \left( \frac{f \cdot x_Q}{C_2^{2/\beta} \lambda} \right) \Phi \left( \frac{1}{t} \right) \left( \frac{f \cdot x_Q}{C_2^{2/\beta} \lambda} \right) \Phi \left( \frac{1}{t} \right) \left( \frac{f \cdot x_Q}{\lambda} \right)^\eta \, ds$$

$$\leq \frac{1}{t}.$$  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
provided \((f \cdot \chi_Q)^* (|Q| t) > \lambda/2\). Thus,
\[
\frac{1}{2} \leq \int_{\{t: (f \cdot \chi_Q)^* (|Q| t) > \lambda/2\}} \frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \phi \left( \frac{1}{t} \right) \, dt \\
\leq \int_{\{t: (f \cdot \chi_Q)^* (|Q| t) > \lambda/2\}} \frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \phi \left[ \left( \frac{(f \cdot \chi_Q)^* (|Q| t)}{C_2 \lambda} \right)^{\beta} \right] \, dt \\
= \frac{1}{|Q|} \int_{Q \cap \{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \phi \left[ \left( \frac{|f(x)|}{C_2 \lambda} \right)^{\beta} \right] \, dx.
\]

This completes the proof of Lemma 1.

Before continuing we would like to make an observation. If a specific nondecreasing \(\phi\) was under consideration, a better estimate could be obtained in Lemma 1 by using, for instance, \(|t \log^2 [2/t]|^{-1}\) instead of \(t^{-\delta}\) in the proof. This would result in a sharper, though more complicated, inequality in Theorem 1. Not much is gained if \(\phi\) is a logarithmic function.

The next lemma contains the converse inequalities to Lemma 1 that will be used to prove Theorem 2. We follow convention and define
\[
\phi^+ (|f(x)|) = \phi (\max \{1, |f(x)|\}).
\]

**Lemma 2.** Let \(\lambda > 0\) and \(Q \subset \mathbb{R}^n\). If \(\phi\) is nonincreasing and \(\eta > 1/(2 - \alpha)\), then there is a \(C_\eta\) such that
\[
|Q| \leq \int_Q \frac{|f(x)|}{\lambda} \phi^+ \left[ \left( \frac{\phi (|f(x)|)}{\lambda} \right)^{\eta} \right] \, dx
\]
implies
\[
\frac{1}{4} \leq \int_0^1 \frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \phi \left( \frac{1}{t} \right) \, dt.
\]
If \(\phi\) is nondecreasing and
\[
|Q| \leq \int_Q \frac{|f(x)|}{\lambda} \phi^+ \left[ \left( \frac{|f(x)|}{\lambda} \right)^{1/(2 - \beta)} \right] \, dx,
\]
then there is a constant \(C\) such that
\[
1 \leq C \int_0^1 \frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \phi \left( \frac{1}{t} \right) \, dt.
\]

**Proof.** We first assume \(\phi\) is nondecreasing. Let \(B = \max \{1, \phi (1)\}\) and \(C_2\) be the constant, where \(t \phi (t) \geq C_2 t^\beta, 1 \leq t < \infty\). We may assume
\[
\int_0^1 \frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \phi \left( \frac{1}{t} \right) \, dt < \frac{C_2}{2B^*}.
\]
If not, we are done. Using the lower bound for \(t \phi (t)\) we estimate
\[
\frac{(f \cdot \chi_Q)^* (|Q| t)}{\lambda} \leq \frac{1}{t \phi (1/t)} \int_0^t \frac{(f \cdot \chi_Q)^* (|Q| s)}{\lambda} \phi \left( \frac{1}{s} \right) \, ds \\
\leq \frac{1}{2Bt^{2-\beta}}.
\]
Thus,

$$1 \leq \int_0^1 \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t)_\Phi + \left( \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t) \right)^{1/(2-\beta)} dt$$

implies

$$\frac{1}{2} \leq \int_{\{t: (f \cdot \chi_Q)^\ast(|Q|t) > \lambda/2B\}} \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t)_\Phi \left( \left( \frac{2B (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^{1/(2-\beta)} \right) dt$$

$$\leq \int_0^1 \frac{(f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \Phi \left( \frac{1}{t} \right) dt.$$ 

Now let $\Phi$ be nonincreasing. We require $C_n > 2B$. Then

$$1 \leq \int_0^1 \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t)_\Phi + \left( \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t) \right)^\eta dt$$

implies

$$\frac{1}{2} \leq \int_{\{t: (f \cdot \chi_Q)^\ast(|Q|t) > \lambda/2B\}} \left( \frac{f \cdot \chi_Q}{\lambda} \right)^\ast(|Q|t)_\Phi \left( \left( \frac{C_n (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^\eta \right) dt.$$ 

For $0 < \delta < 1$ let

$$J_\delta = \left\{ t \in (0,1]: \frac{(f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \Phi \left( \left( \frac{C_n (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^\eta \right) \leq \frac{(1-\delta)}{4 \cdot t^{\delta}} \right\}.$$ 

Then

$$\frac{1}{4} \leq \int_{\{t: (f \cdot \chi_Q)^\ast(|Q|t) > \lambda/2B\} \setminus J_\delta} \frac{(f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \Phi \left( \left( \frac{C_n (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^\eta \right) dt.$$ 

For $t \in \{t: (f \cdot \chi_Q)^\ast(|Q|t) > \lambda/2B\} \setminus J_\delta$ we have, using $C_1 t^{-\alpha} \geq t^{-1} \Phi(t^{-1})$ and the above,

$$\frac{C_1}{C_n} \left[ \left( \frac{C_n (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^{1-\eta+\alpha \eta} \right] \geq \frac{1}{4} \frac{(1-\delta)}{t^{\delta}}.$$ 

Hence, if we require $\eta/(1-\eta+\alpha \eta) > 1$, i.e., $\eta > 1/(2-\alpha)$, we may choose $\delta = (1-\eta+\alpha \eta)/\eta < 1$ and $C_n = \max\{2B, 4C_1/(1-\delta)\}$ to obtain

$$\left( \frac{C_n (f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \right)^\eta \geq \frac{1}{t}.$$ 

Thus,

$$\frac{1}{4} \leq \int_0^1 \frac{(f \cdot \chi_Q)^\ast(|Q|t)}{\lambda} \Phi \left( \frac{1}{t} \right) dt$$

and the proof of Lemma 2 is complete.

We now show the maximal $M_\Phi f$ satisfies a weak type inequality. The results are listed as Theorem 1.

**Theorem 1.** Let $\lambda > 0$. If $\Phi$ is nonincreasing there are constants $A$ and $C > 2$ such that

$$|\{x \in \mathbb{R}^n: M_\Phi f(x) > \lambda\}| \leq A \int_{\{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \Phi \left( \left( \frac{C |f(x)|}{\lambda} \right)^\beta \right) dx.$$
If $\Phi$ is nondecreasing there are constants $B_\eta$ and $C_\eta > 2$ such that, for $\eta > 1/(2-\alpha)$,

$$|\{x \in \mathbb{R}^n : M_\Phi f(x) > \lambda\}| \leq B_\eta \int_{\{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \Phi \left[ \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \right] dx.$$ 

**Proof.** We let

$$M_r f(x) = \sup_{0 < r} \int_0^1 (f \cdot \chi_Q)^*(|Q|t) \Phi \left( \frac{1}{t} \right) dt,$$

where the sup is extended over all $Q$ with center $x$ and diam $Q \leq r$. It suffices to prove the theorem for $M_r f$ and then let $r \to \infty$.

Let $E_\lambda = \{x : M_r f(x) > \lambda\}$ and $E_{\lambda,R} = E_\lambda \cap \{|x| < R\}$. For $x \in E_{\lambda,R}$, we have a cube $Q_x$, with center $x$, and diam $Q_x \leq r$ such that

$$\lambda \leq \int_0^1 (f \cdot \chi_Q)^*(|Q|t) \Phi \left( \frac{1}{t} \right) dt.$$ 

We can now apply the Besicovitch covering theorem [1] and select $\{Q_j\} \subset \{Q_x : x \in E_{\lambda,R}\}$ such that $E_{\lambda,R} \subset \bigcup Q_j$ and $\sum \chi_{Q_j} < C$, where $C$ depends only upon $n$. If we now assume $\Phi$ is nondecreasing, then by Lemma 1 we have

$$|E_{\lambda,R}| \leq \sum |Q_j| \leq \sum C_\eta \int_{Q_j \cap \{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \Phi \left[ \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \right] dx \quad (\eta > \frac{1}{2-\alpha})$$

$$\leq B_\eta \int_{\{|f| > \lambda/2\}} \frac{|f(x)|}{\lambda} \Phi \left[ \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \right] dx.$$ 

We now let $R \not\to \infty$ and then $r \not\to \infty$ to complete the proof. The case where $\Phi$ is nondecreasing uses Lemma 1 and the proof is the same.

We now show that the maximal operator $M$ satisfies an inequality that is related to the converse of the corresponding weak type result of Theorem 1.

**Theorem 2.** Let $\lambda > 0$. If $\Phi$ is nonincreasing we have, for some constants $A$ and $C_\eta$,

$$|\{x \in \mathbb{R}^n : M_{\eta} f(x) > \lambda A\}| \geq A \int_{\{|f| > \lambda A\}} \frac{|f(x)|}{\lambda} \Phi^+ \left[ \left( C_\eta \frac{|f(x)|}{\lambda} \right)^\eta \right] dx,$$

where $\eta > 1/(2-\alpha)$.

If $\Phi$ is nondecreasing we have, for some constant $B$,

$$|\{x \in \mathbb{R}^n : M_{\eta} f(x) > \lambda B\}| \geq B \int_{\{|f| > \lambda B\}} \frac{|f(x)|}{\lambda} \Phi^+ \left[ \left( \frac{|f(x)|}{\lambda} \right)^{1/(2-\beta)} \right] dx.$$ 

**Proof.** We will just do the nonincreasing case, noting that the nondecreasing case is similar. For each of notation we let $\Phi_1(t) = \Phi[(C_\eta t)^\eta]$ for a fixed $\eta > 1/(2-\alpha)$. Let

$$\bar{f}(x) = \min\{r, |f(x)| \cdot \chi_{B(0,r)}(x)\}.$$

Since $M_{\eta} f(x) \geq M_{\eta} \bar{f}(x)$, it suffices to prove the theorem for $\bar{f}(x)$ and then let $r \not\to \infty$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
We apply the Calderon-Zygmund Lemma \cite{6} to decompose \( \mathbb{R}^n \) into sets \( E \) and \( F \) where

(i) \( F \cap E = \emptyset \),

(ii) \( \left( \frac{|f(x)|}{\lambda} \right)^+ \left( \frac{|f(x)|}{\lambda} \right) \leq 2^n \), a.e. \( x \in F \),

(iii) \( E = \bigcup Q_j \), where \( Q_j \cap Q_k = \emptyset \) for \( j \neq k \) and

\[
2^n \leq \frac{1}{|Q_j|} \int_{Q_j} \left( \frac{|f(x)|}{\lambda} \right)^+ \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 2^{2n}.
\]

We note that (ii) implies \( |f(x)| \leq 2^n \lambda / \Phi(1) \) for a.e. \( x \in F \). For a fixed \( x \in Q_j \) let \( Q_x \) be the smallest cube centered at \( x \) containing \( Q_j \). Then the above implies

\[
1 \leq \frac{1}{|Q_x|} \int_{Q_x} \left( \frac{|f(x)|}{\lambda} \right)^+ \left( \frac{|f(x)|}{\lambda} \right) \, dx.
\]

Hence, by Lemma 2, we have \( M_{\Phi} f(x) \geq \lambda / 4 \) for every \( x \in Q_j \). Summing over \( Q_j \) and letting \( A = \min\{2^{-2n}, 2^{-n} \Phi(1)\} \) completes the proof.

3. Natural and interesting examples arise when \( \Phi \) is a logarithmic type of function. In such a case Theorems 1 and 2 assume an elegant form since the exponents do not play a significant role.

Consider the following formula for the \( N \)th iteration of the Hardy-Littlewood maximal operator (see \cite[p. 5]{5}):

\[
M \cdots M f(x) \sim \sup_{x \in Q} \int_0^1 (f \cdot \chi_Q)'(t) \frac{\log^{N-1}(1/t)}{(N-1)!} \, dt.
\]

We generalize the above formula for \( -\infty < a < \infty \) as follows. For a real let

\[
M_a f(x) = \sup_{x \in Q} \int_0^1 (f \cdot \chi_Q)'(t) \frac{\log^a(e/t)}{C(a)} \, dt,
\]

where \( C(a) = \int_0^1 \log^a(e/t) \, dt \). We note that \( e/t \) avoids the strong singularity at \( t = 1 \) if \( a \leq -1 \). For this maximal operator, Theorems 1 and 2 assume a recognizable form which we state as Theorem 3.

**THEOREM 3.** Let \( \lambda > 0 \) and \( a \) be a given fixed number. Then there exist constants \( A \) and \( B \) which depend only on dimension \( N \) and \( a \) such that

\[
\{x \in \mathbb{R}^n : M_a f(x) > \lambda \} \leq A \int_{\{|f| > \lambda / A\}} \frac{|f(x)|}{\lambda} \left[ 1 + \log^+ \left( \frac{|f(x)|}{\lambda} \right) \right]^a \, dx,
\]

\[
\{x \in \mathbb{R}^n : M_a f(x) < \lambda \} \geq B \int_{\{|f| > \lambda / B\}} \frac{|f(x)|}{\lambda} \left[ 1 + \log^+ \left( \frac{|f(x)|}{\lambda} \right) \right]^a \, dx.
\]

**REFERENCES**


License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use


**DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, FLORIDA 33199**