PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 1, September 1984

ON DISCONJUGACY AND k-POINT DISCONJUGACY FOR LINEAR ORDINARY DIFFERENTIAL OPERATORS

JAMES S. MULDOWNEY[1]

ABSTRACT. It is shown that a linear ordinary differential operator of order n which is k-point disconjugate on an open interval for some k, $2 \leq k \leq n$, is also disconjugate on that interval.

1. Introduction. Consider the linear differential operator L of order n defined by

$$Lu = u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_n(t)u$$

where a_i, $i = 1, \ldots, n$, are continuous real valued functions on an open interval I. The operator L is said to be *disconjugate* on an interval $J \subset I$ if the only solution of the equation $Lu = 0$ having n zeros or more in J, counting multiplicities, is the zero solution. The operator is said to be *k-point disconjugate* on J if the zero solution is the only solution of $Lu = 0$ having zeros at k distinct points in J, the sum of whose multiplicities is not less than n.

It is clear that, for any interval J, disconjugacy implies k-point disconjugacy. Conversely, for open intervals, it is a special case of a result of Hartman [**1**] that n-point disconjugacy implies disconjugacy and Sherman [**4**] showed that 2-point disconjugacy implies disconjugacy for any interval. The question of whether k-point disconjugacy ($2 < k < n$) implies disconjugacy on open intervals has been addressed by Henderson [**2**] who showed that this implication holds subject to strong additional restrictions on k. This paper removes these restrictions.

2. Result. We prove the following theorem.

THEOREM 2.1. *Suppose $2 \leq k \leq n$ and L is k-point disconjugate on the open interval I. Then L is disconjugate on I.*

The proof will be furnished by introducing some notation and establishing the validity of a number of assertions. In view of the result of Sherman [**4**], it is sufficient to prove the theorem for $2 < k \leq n$.

Let $u_1, \ldots, u_n$ be a fundamental solution set for the equation $Lu = 0$ such that the wronskian determinant $W(u_1, \ldots, u_n) = \det[u_i^{(j-1)}]$ is positive on I. If $\{\tau_1, \ldots, \tau_k\} \subset I$ and $m_1, \ldots, m_k$ are nonnegative integers such that $m_1 + \cdots + m_k = n$, let $\mathcal{W}(\tau_1, m_1; \tau_2, m_2; \ldots; \tau_k, m_k)$ denote the $n \times n$ matrix of which the ith column, $i = 1, \ldots, n$, is

$$\operatorname{col}[u_i(\tau_1), \ldots, u_i^{(m_1-1)}(\tau_1), u_i(\tau_2), \ldots, u_i^{(m_2-1)}(\tau_2), \ldots, u_i(\tau_k), \ldots, u_i^{(m_k-1)}(\tau_k)].$$

Received by the editors July 20, 1983.
1980 *Mathematics Subject Classification*. Primary 34C10.
Key words and phrases. Linear differential equation, disconjugacy.
[1]Research supported by NSERC grant 7197.

If $m_j = 0$ there are no terms corresponding to the point τ_j. Note that $\det \mathcal{W}(\tau, n) = W(u_1, \ldots, u_n)(\tau)$.

Suppose that L is not disconjugate on I. Then there exist points $\alpha, \gamma \in I$ such that γ is the first right conjugate point of α, which means L is not disconjugate on $[\alpha, \gamma]$, but is disconjugate on every proper subinterval of $[\alpha, \gamma]$. Let $\beta \in (\alpha, \gamma)$.

ASSERTION 2.2. *There exist integers p, q, r with $p > 0,\ q \geq 0,\ r > 0,\ p+q+r = n$ such that*

$$\text{rank}\, \mathcal{W}(\alpha, p; \beta, q; \gamma, r) < n. \tag{1}$$

Assertion 2.2 is equivalent to the existence of a nonzero solution of $Lu = 0$ having p zeros at α, q zeros at β and r zeros at γ. The existence of such a solution was established by Sherman [**4**] with $q = 0$.

ASSERTION 2.3 *First choose q to be the largest integer such that* (1) *holds. Next let r be maximal such that* (1) *holds for this choice of q and let $p = n - q - r$.*

(a) *At least one of the cofactors of the last row,* $\text{row}[u_1^{(r-1)}(\gamma), \ldots, u_n^{(r-1)}(\gamma)]$, *in* $\mathcal{W}(\alpha, p; \beta, q; \gamma, r)$ *is nonzero.*

(b) *There is a solution u of $Lu = 0$ which has p zeros at α, q zeros at β and r zeros at γ, $u^{(r)}(\gamma) \neq 0$, and which is unique to within a constant multiple.*

To prove Assertion 2.3(a), observe that, if these cofactors were all zero, this would imply

$$\text{rank}\ \mathcal{W}(\alpha, p; \beta, q+1; \gamma, r-1) < n,$$

contradicting the maximal character of q if $r - 1 > 0$ and contradicting the disconjugacy of L on $[\alpha, \beta]$ if $r - 1 = 0$.

Assertion 2.3(b) follows from the fact that (1) implies the existence of a nontrivial solution u of $Lu = 0$ which has p zeros at α, q zeros at β and r zeros at γ. If $u^{(r)}(\gamma) = 0$ for any such solution, then

$$\text{rank}\, \mathcal{W}(\alpha, p-1; \beta, q; \gamma, r+1) < n. \tag{2}$$

But, if $p - 1 > 0$, (2) contradicts the maximality of r; if $p - 1 = 0,\ q > 0$, (2) contradicts the disconjugacy of L on $[\beta, \gamma]$; if $p - 1 = 0,\ q = 0$, (2) contradicts $W(u_1, \ldots, u_n)(\gamma) > 0$. Thus $u^{(r)}(\gamma) \neq 0$. To see that u is unique, note that the existence of two independent solutions with p zeros at α, q zeros at β and r zeros at γ would imply the existence of such a solution with $u^{(r)}(\gamma) = 0$ which, as we have proved, leads to a contradiction.

ASSERTION 2.4. *Let p, q, r be chosen as in Assertion* 2.3. *Then the function*

$$F(t) = \det \mathcal{W}(\alpha, p; \beta, q; t, r)$$

changes sign at $t = \gamma$.

We prove this assertion by considering the function

$$u(t) = \det \mathcal{W}(\alpha, p; \beta, q; \gamma, r-1; t, 1),$$

which is a solution of $Lu = 0$. It is a nontrivial solution, by Assertion 2.3(a), and clearly

$$\begin{aligned} u^{(i-1)}(\alpha) &= 0, \qquad i = 1, \ldots, p, \\ u^{(i-1)}(\beta) &= 0, \qquad i = 1, \ldots, q, \\ u^{(i-1)}(\gamma) &= 0, \qquad i = 1, \ldots, r-1; \end{aligned}$$

also $u^{(r-1)}(\gamma) = \det \mathcal{W}(\alpha, p; \beta, q; \gamma, r) = 0$, from (1). Thus u is the unique solution of $Lu = 0$ whose existence is Assertion 2.3(b). Now

$$F(\gamma) = \det \mathcal{W}(\alpha, p; \beta, q; \gamma, r) = 0,$$

from (1), and $F'(\gamma) = u^{(r)}(\gamma) \neq 0$, from Assertion 2.3(b). Therefore F changes sign at γ.

ASSERTION 2.5. *Let $2 < k \leq n$ and let F be the function defined in Assertion 2.4. If L is k-point disconjugate on I, then $F(t) \geq 0$ for all $t > \beta$, $t \in I$.*

The k-point disconjugacy of L on I implies that

$$\det \mathcal{W}(\tau_1, m_1; \ldots; \tau_k, m_k) \neq 0 \tag{3}$$

if $m_i > 0$, $\tau_i \in I$, $i = 1, \ldots, k$, $\tau_1 < \tau_2 < \cdots < \tau_k$. Continuity in $(\tau_1, \ldots, \tau_k)$ implies that, for any choice of $(m_1, \ldots, m_k)$, all of the determinants considered in (3) are of one sign. They are in fact positive since, by successive applications of the Mean Value Theorem, $\det \mathcal{W}(\tau_1, m_1; \cdots; \tau_k, m_k)$ is a positive multiple of a determinant of the form $\det[u_i^{(j-1)}(\alpha_j)]$, $i, j = 1, \ldots, n$, where $\tau_1 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n < \tau_k$. When $\tau_1, \ldots, \tau_k$ are all close to a point τ, then so also are $\alpha_1, \ldots, \alpha_n$ and $\det \mathcal{W}(\tau_1, m_1; \cdots; \tau_k, m_k) > 0$ follows from continuity and $W(u_1, \ldots, u_n)(\tau) > 0$. Next, choose $m_1, \ldots, m_k$, h, j such that $m_i > 0$, $i = 1, \ldots, k$, $1 \leq h \leq j < k$ and

$$m_1 + \cdots + m_h = p, \quad m_{h+1} + \cdots + m_j = q, \quad m_{j+1} + \cdots + m_k = r.$$

The case $h = j$ corresponds to $q = 0$ and, in this case, there are no terms $m_{h+1}, \ldots, m_j$ present. Again by Rolle's Theorem, if $\tau_1 < \tau_2 < \cdots < \tau_k$, the positive determinant $\det \mathcal{W}(\tau_1, m_1; \cdots; \tau_k, m_k)$ is a positive multiple of the determinant $\Delta(\alpha_1, \ldots, \alpha_p,\ \beta_1, \ldots, \beta_q,\ t_1, \ldots, t_r)$ of a $n \times n$ matrix of which the ith column has the form

$$\operatorname{col}[u_i(\alpha_1), \ldots, u_i^{(p-1)}(\alpha_p), u_i(\beta_1), \ldots, u_u^{(q-1)}(\beta_q), u_i(t_1), \ldots, u_i^{(r-1)}(t_r)],$$

$i = 1, \ldots, n$, for some numbers $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$, $t_1, \ldots, t_r$ such that

$$\begin{aligned} \tau_1 &= \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p \leq \tau_h, \\ \tau_{h+1} &= \beta_1 \leq \beta_2 \leq \cdots \leq \beta_q \leq \tau_j, \\ \tau_{j+1} &= t_1 \leq t_2 \leq \cdots \leq t_r \leq t_k, \end{aligned} \tag{4}$$

and therefore

$$\Delta(\alpha_1, \ldots, \alpha_p,\ \beta_1, \ldots, \beta_q,\ t_1, \ldots, t_r) > 0 \tag{5}$$

for these numbers. But, if

$$(\alpha_1, \ldots, \alpha_p) \to (\alpha, \ldots, \alpha), \quad (\beta_1, \ldots, \beta_q) \to (\beta, \ldots, \beta), \quad (t_1, \ldots, t_r) \to (t, \ldots, t),$$

then

$$\Delta(\alpha_1, \ldots, \alpha_p,\ \beta_1, \ldots, \beta_q,\ t_1, \ldots, t_r) \to \det \mathcal{W}(\alpha, p; \beta, q; t, r) = F(t).$$

Thus, by considering $\tau_1, \ldots, \tau_k$ close to α, $\tau_{h+1}, \ldots, \tau_j$ close to β and $\tau_{j+1}, \ldots, \tau_k$ close to t, we see from (4) and (5) that $F(t) \geq 0$ if $t > \beta$, proving Assertion 2.5.

Considering Assertions 2.4 and 2.5, we see that the open interval I cannot contain a point α and its first right conjugate point γ while L is k-point disconjugate on I. This proves Theorem 2.1.

3. Remarks. An examination of the proof of Assertion 2.5 shows that Theorem 2.1 is still true if a more stringent definition of k-point disconjugacy is adopted. Specifically, *if* $2 \leq k \leq n$ *and, for each set of points* $t_1, \ldots, t_k$ *in* I *with* $t_1 < t_2 < \cdots < t_k$, $u = 0$ *is the only solution of* $Lu = 0$ *which has a zero of multiplicity* $m_i > 0$ *at* t_i, $i = 1, \ldots, k$, *where* $m_1 + m_k \geq n - k + 2$, *then* L *is disconjugate on* I. In the case $k = 2$, this is again implied by Sherman's result [**4**] and, when $2 < k \leq n$, Assertion 2.5 may be proved with $m_2 = m_3 = \cdots = m_{k-1} = 1$.

With minor modifications, the proof given here may be adapted to provide similar results in the case of disfocality and generalizations thereof; cf. [**3**].

ACKNOWLEDGEMENT. This paper was motivated by conversations with Uri Elias.

REFERENCES

1. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.
2. J. Henderson, *k-point disconjugacy and disconjugacy for linear differential equations*, Univ. of Missouri (preprint 1982).
3. J. S. Muldowney, *On invertibility of linear ordinary differential boundary value problems*, SIAM J. Math. Anal. **12** (1981), 368–384.
4. T. L. Sherman, *Properties of solutions of nth order linear differential equations*, Pacific J. Math. **15** (1965), 1045–1060.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA, T6G 2G1